Sampling normalizing constants in high dimensions using inhomogeneous diffusions

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Abstract

Motivated by the task of computing normalizing constants and importance sampling in high dimensions, we study the dimension dependence of fluctuations for additive functionals of time-inhomogeneous Langevin-type diffusions on $\mathbb{R}^d$. The main results are nonasymptotic variance and bias bounds, and a central limit theorem in the $d \to \infty$ regime. We demonstrate that a temporal discretization inherits the fluctuation properties of the underlying diffusion, which are controlled at a computational cost growing at most polynomially with d. The key technical steps include establishing Poincaré inequalities for time-marginal distributions of the diffusion and nonasymptotic bounds on deviation from Gaussianity in a martingale central limit theorem.

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1 Introduction

Consider $(X_t^\epsilon)_{t\in[0,1]}$ the time-inhomogeneous diffusion on $\mathbb{R}^d$ which solves

$$X_t^\epsilon = X_0^\epsilon - \epsilon^{-1}\int_0^t \nabla U_s(X_s^\epsilon)\mathrm{d}s + \sqrt{2\epsilon^{-1}}\int_0^t \mathrm{d}B_s, \tag{1}$$

where B_t is d-dimensional Brownian motion, $\epsilon > 0$ is a parameter and $(U_t)_{t\in[0,1]}$ is a family of $\mathbb{R}$-valued potentials such that, with Lebesgue measure and the Borel σ-algebra denoted by $\mathrm{d}x$ and $\mathcal{B}(\mathbb{R}^d)$, $(\pi_t)_{t\in[0,1]}$ given by

$$Z_t := \int_{\mathbb{R}^d} \exp\{-U_t(x)\}\mathrm{d}x, \qquad \pi_t(A) := Z_t^{-1}\int_A \exp\{-U_t(x)\}\mathrm{d}x, \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{2}$$

are well-defined as probability measures.

This work concerns dependence on the dimension, d, of fluctuations associated with

$$S_\epsilon := \int_0^1 f_t(X_t^\epsilon)\mathrm{d}t, \qquad S_{\epsilon,h} := h\sum_{k=0}^{\lfloor 1/h\rfloor -1} f_{kh}(X_{kh}^\epsilon), \qquad \tilde{S}_{\epsilon,h} := h\sum_{k=0}^{\lfloor 1/h\rfloor -1} f_{kh}(\tilde{X}_{kh}^{\epsilon,h}), \tag{3}$$

where $(f_t)_{t\in[0,1]}$ is a family of $\mathbb{R}$-valued functions such that each f_t is centred with respect to π_t, and $(\tilde{X}_t^{\epsilon,h})_{t\in[0,1]}$ is an approximation to $(X_t^\epsilon)_{t\in[0,1]}$ such that the skeleton variables $\tilde{X}_{kh}^{\epsilon,h}$ can be simulated by a time-discretization method, and $h \in (0,1]$ is a step-size parameter such that the cost of the discretization scheme is proportional to h^{-1}.

Amongst our key assumptions, which we state precisely later, will be strong convexity in x of $U_t(x)$, or equivalently strong log-concavity of π_t. As accounted in [1], thorough investigations have been made of the connections between concentration of measure phenomena, Poincaré and other functional inequalities for log-concave measures and the ergodic properties of time-homogeneous Markov processes, such as the diffusion in (1) in the case that U_t does not depend on t. These connections have been exploited to study the computational cost of approximate sampling from log-concave measures using *Markov chain Monte Carlo* (MCMC) algorithms, via bounds on distance to equilbrium and error estimates for ergodic averages which elicit dependence on dimension, e.g. [14, 13, 21, 8].

Our primary motivation for studying the time-inhomogeneous case is connected with another Monte Carlo technique: *importance sampling,* which along with MCMC is one of the most popular simulation-based methods for numerical integration, and is applied across scientific disciplines such as statistical physics, signal processing and machine learning. Although as we shall illustrate next, importance sampling in its most basic form can perform exponentially badly in high dimensions, one of the main insights which can be drawn from our results is that a more sophisticated type of importance sampling technique using an inhomogeneous Markov process can be practically reliable, in a sense which we shall make precise, at a cost polynomial in d.

1.1 Motivation: importance sampling and thermodynamic integration

As an elementary example to illustrate the motivating ideas, consider the task of numerically approximating the ratio of normalizing constants Z_1/Z_0 and the expectation $\pi_1(f) := \int_{\mathbb{R}^d}\varphi(x)\pi_1(\mathrm{d}x)$ for some test function φ, assuming that one is only able to simulate $(\zeta_1,\ldots,\zeta_m) \overset{\text{i.i.d.}}{\sim} \pi_0$ and evaluate U_0, U_1 and φ pointwise. With $W_i := \exp[-\{U_1(\zeta_i) - U_0(\zeta_i)\}]$, so

$$\frac{Z_1}{Z_0} = \mathbb{E}[W_i], \qquad \pi_1(\varphi) = \frac{\mathbb{E}[\varphi(\zeta_i)W_i]}{\mathbb{E}[W_i]},$$

the basic importance sampling method reports the approximations:

$$\frac{Z_1}{Z_0} \approx \frac{1}{m}\sum_{i=1}^m W_i, \qquad \pi_1(\varphi) \approx \frac{\sum_{i=1}^m \varphi(\zeta_i)W_i}{\sum_{i=1}^m W_i}. \tag{4}$$

If for sake of illustration the potentials are of the form:

$$U_t(x) = \sum_{j=1}^d u_t(x^j), \quad x = (x^1,\ldots,x^d), \tag{5}$$

we have for any i,

$$\frac{\operatorname{var}[W_i]}{\mathbb{E}[W_i]^2} = c^d - 1, \tag{6}$$

where $c := \mathbb{E}[\exp -2\{u_1(\zeta_1^1) - u_0(\zeta_1^1)\}]/\mathbb{E}[\exp -\{u_1(\zeta_1^1) - u_0(\zeta_1^1)\}]^2$ does not depend on d, and ζ_1^1 is the first of the d co–ordinates of ζ_1. By Jensen's inequality $c \geqslant 1$ with equality if and only if $\pi_1 = \pi_0$, so putting aside that trivial case, (6) indicates that the cost of the simulation, governed by m, must be increased exponentially in d in order to prevent growth of the relative errors associated with (4). Also when $c > 1$, the total variation distance between π_0 and π_1 is monotonically increasing in d, and indeed as d reaches infinity, π_0 and π_1 become singular in the sense of Kakutani's theorem on infinite product measures. Intuitively the "one-step" importance sampling correction from π_0 to π_1 in (4) is defeated by this phenomenon.

An alternative approach is based around the representation formulae:

$$\frac{Z_1}{Z_0} = \exp\left\{-\int_0^1 \pi_t(\partial_t U_t)\mathrm{d}t\right\} = \mathbb{E}\left[\exp\left\{-\int_0^1 \partial_t U_t(X_t^\epsilon)\mathrm{d}t\right\}\right], \tag{7}$$

$$\pi_1(\varphi) = \frac{\mathbb{E}\left[\varphi(X_1^\epsilon)\exp\left\{-\int_0^1 \partial_t U_t(X_t^\epsilon)\mathrm{d}t\right\}\right]}{\mathbb{E}\left[\exp\left\{-\int_0^1 \partial_t U_t(X_t^\epsilon)\mathrm{d}t\right\}\right]}, \tag{8}$$

where $(X_t^\epsilon)_{t\in[0,1]}$ as in (1) with any $\epsilon > 0$ and $X_0^\epsilon \sim \pi_0$, and $\partial_t U_t$ is the partial derivative of U_t w.r.t. t, and $\pi_t(\partial_t U_t)$ is the integral with respect to π_t (we shall later discuss technical assumptions under which validity of (7)–(8) can be rigorously established). The equalities in (7) have roots in the statistical physics literature, the first being known as the thermodynamic integration or path sampling identity, see [15] for an account of its history, the second as Jarzynki's equality [20, 19]. The expectations in (7)–(8) have an importance sampling interpretation: $\exp\left\{-\int_0^1 \partial_t U_t(X_t^\epsilon)\mathrm{d}t\right\}\frac{Z_0}{Z_1}$ can be derived as the Radon-Nikodym derivative, with respect to the path measure of $(X_t^\epsilon)_{t\in[0,1]}$, of the law of a diffusion whose drift is transformed such that its marginal distribution at time $t = 1$ is π_1, see [33, Section 3.2, p.62] for a time-reversal perspective and [32, Ch. VIII, Sec. 3] for background on this type of transformation. The discrete-time counterpart of (8) is the basis for the Annealed Importance Sampling method of [28].

In light of (7)–(8), an alternative to the basic importance sampling method described above is obtained by replacing each pair $W_i, \varphi(\zeta_i)$ in (4) with an independent copy of the pair $\exp\left\{-\int_0^1 \partial_t U_t(X_t^\epsilon)dt\right\}, \varphi(X_1^\epsilon)$, or in practice some approximation thereof involving time-discretization. If in (3) one takes $f_t(x) = \partial_t U_t(x) - \pi_t(\partial_t U_t)$, then from (7),

$$S_\epsilon = \int_0^1 \partial_t U_t(X_t^\epsilon) - \pi_t(\partial_t U_t)\mathrm{d}t = \int_0^1 \partial_t U_t(X_t^\epsilon)\mathrm{d}t - \log\frac{Z_0}{Z_1},$$

hence our interest in the dimension dependence of the fluctuations associated with (3).

To see why there is hope that this scheme can perform well in high dimensions, note that in the setting (5) the co-ordinates $(X_t^{\epsilon,1}, \ldots, X_t^{\epsilon,d})$ of X_t^ϵ are i.i.d., as are the summands in:

$$S_\epsilon = \sum_{j=1}^d \int_0^1 \partial_t u_t(X_t^{\epsilon,j}) - \pi_t(\partial_t u_t)\mathrm{d}t,$$

where $\pi_t(\partial_t u_t)$ is the integral of $\partial_t u_t$ w.r.t. any of the 1-dimensional marginals of π_t. So, if $\int_0^1 \partial_t u_t(X_t^{\epsilon,j}) - \pi_t(\partial_t u_t)\mathrm{d}t$ is of order $O_p(\epsilon^{1/2})$ as $\epsilon \to 0$, and ϵ is chosen to be d^{-2}, then S_ϵ is of order $O_p(1)$ as $d \to \infty$. If also $\sum_{j=1}^d \int_0^1 \partial_t u_t(X_t^{\epsilon,j})$ can be well-approximated by discretization at a cost proportional to h^{-1} and polynomial in ϵ^{-1}, then overall one obtains a method to approximate (7)–(8) which does not suffer from exponentially bad behaviour in high dimensions.

Of course in situations of practical interest, each π_t is usually not a product measure, i.e. U_t is not of the form in (5), and the dependence on d of the fluctuations of S_ϵ in such situations is a less simple matter. Discussion of our approach is given after introducing notation and assumptions.

1.2 Notation

Inner-product and Euclidean norm on $\mathbb{R}^d$ are denoted by respectively $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. The $d\times d$ zero and identity matrices are written 0_d and I_d, and e_i denotes the vector in $\mathbb{R}^d$ whose i'th entry is 1 and whose other entries are zeros. For a q-dimensional array A with real entries $A[i_1,\cdots,i_q]=a_{i_1,\cdots,i_q}$, $(i_1,\ldots,i_q)\in\{1,\ldots,d\}^q$, the Hilbert-Schmidt norm is denoted $\|A\|_{\mathrm{H.S.}}:=\left(\sum_{(i_1,\ldots,i_q)\in\{1,\ldots,d\}^q}a^2_{i_1,\cdots,i_q}\right)^{1/2}$. When such an array depends on an argument $x\in\mathbb{R}^d$, we define for $p\geqslant 1$,

$$\|A\|_p := \sup_{x\in\mathbb{R}^d}\frac{\|A(x)\|_{\mathrm{H.S.}}}{1+\|x\|^{2p}}. \tag{9}$$

For a function $f:\mathbb{R}^d\to\mathbb{R}$, we write $\nabla^{(q)}f$ for the q-dimensional array of q-th order partial derivatives of f, with entries $\nabla^{(q)}f[i_1,\ldots,i_q]=\frac{\partial^q f}{\partial x_{i_1}\cdots\partial x_{i_q}}$, where $(i_1,\ldots,i_q)\in\{1,\ldots,d\}^q$. In particular the usual gradient is $\nabla^{(1)}\equiv\nabla$ and by convention we take $\nabla^{(0)}f\equiv f$. The Laplacian operator is denoted Δ. As instances of (9) we have for example,

$$\|f\|_p=\sup_{x\in\mathbb{R}^d}\frac{|f(x)|}{1+\|x\|^{2p}},\qquad \|\nabla^{(q)}f\|_p=\sup_{x\in\mathbb{R}^d}\frac{\|\nabla^{(q)}f(x)\|_{\mathrm{H.S.}}}{1+\|x\|^{2p}}. \tag{10}$$

We follow the convention of terminology that a 0-times continuously differentiable function is continuous. For $q\geqslant 0$ and $p\geqslant 1$, let $C^p_q(\mathbb{R}^d)$ be the set of functions $f:\mathbb{R}^d\to\mathbb{R}$ which are q-times continuously differentiable and such that $\|\nabla^{(r)}f\|_p<+\infty$, for $0\leqslant r\leqslant q$.

We shall frequently encounter $\mathbb{R}$-valued functions with domain $[0,1]\times\mathbb{R}^d$ or some subset thereof. For such a function, say $f:(t,x)\in[0,1]\times\mathbb{R}^d\mapsto f(t,x)\in\mathbb{R}$, we shall write interchangeably $f_t(x)\equiv f(t,x)$. With t fixed, we write $\nabla^{(q)}f_t$ for the array of qth-order derivatives of the function $f(t,\cdot):\mathbb{R}^d\to\mathbb{R}$, and with x fixed, we write $\partial_t^q f_t(x)$ for the q-th partial derivative of $f(\cdot,x):[0,1]\mapsto\mathbb{R}$, with $\partial_t^1\equiv\partial_t$. Then $\|\nabla^{(q)}f_t\|_p$ (resp. $\|\partial_t^q f_t\|$) is as in (10) with $\nabla^{(q)}f$ there replaced by $\nabla^{(q)}f_t$ (resp. $\partial_t^q f_t$).

For nonnegative integers q_t,q_x, let $C^p_{q_t,q_x}([0,1]\times\mathbb{R}^d)$ be the set of functions $f:[0,1]\times\mathbb{R}^d\to\mathbb{R}$ such that $f(t,x)$ is q_t-times continuously differentiable in t, q_x-times continously differentiable in x,

$$\sup_{t\in[0,1]}\|\partial_t^r f_t\|_p<+\infty,\quad 0\leqslant r\leqslant q_t,\quad\text{and}\quad \sup_{t\in[0,1]}\|\nabla^{(r)}f_t\|_p<+\infty,\quad 0\leqslant r\leqslant q_x.$$

Define

$$V(x):=\|x\|^2,\qquad \bar V(x):=1+V(x),\qquad \bar V^{(p)}(x):=1+V^p(x),\quad p>0.$$

Below we shall identify for each $t\in[0,1]$ a distinguished point $x_t^\star$, then write $V_t(x):=\|x-x_t^\star\|^2$, $\bar V_t(x):=1+V_t(x)$, $\bar V_t^{(p)}(x):=1+V_t^p(x)$.

The total variation distance between two probabiliity measures ν,ν' on a σ-algebra $\mathcal{G}$ is written $\|\nu-\nu'\|_{\mathrm{tv}}=\sup_{A\in\mathcal{G}}|\nu(A)-\nu'(A)|$. The integral of a function f w.r.t. a measure ν is written νf or $\nu(f)$. The Borel σ-algebra and Lebesgue measure on $\mathbb{R}^d$ are denoted respectively $\mathcal{B}(\mathbb{R}^d)$ and $\mathrm{d}x$. The set of probability measures ν on $\mathcal{B}(\mathbb{R}^d)$ such that $\nu(V^p)<+\infty$ is denoted $\mathcal{P}^p(\mathbb{R}^d)$.

Throughout the paper $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P})$ is a filtered probability space satisfying the usual conditions, on which all the random variables we encounter are defined, and $(B_t)_{t\in\mathbb{R}_+}$ is a d-dimensional $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-Brownian motion. Expectation with respect to $\mathbb{P}$ is denoted $\mathbb{E}$.

With U_t and Z_t as in (2), we denote:

$$\phi_t(x):=-\partial_t U_t(x)-\partial_t\log Z_t. \tag{11}$$

1.3 Assumptions

Fix a function $U:(t,x)\in[0,1]\times\mathbb{R}^d\mapsto U(t,x)\in\mathbb{R}^+$.

(A1) For some $p_0\geqslant 1$, $U\in C^{p_0}_{1,2}([0,1]\times\mathbb{R}^d)$.

(**A2**) (time-uniform Lipschitz gradient) $\exists L < +\infty$ s.t.

$$\sup_{t\in[0,1]} \|\nabla U_t(x) - \nabla U_t(y)\| \leqslant L\|x-y\|, \quad \forall x, y.$$

(**A3**) (regularity in time)

$$\sup_{t\in[0,1]} \|\nabla U_t(x)\| \leqslant L(1+\|x\|), \quad \forall x, \tag{12}$$

where L is as in (A2)

(**A4**) (time-uniform strong convexity) $\exists K > 0$ s.t. $\forall v \in \mathbb{R}^d$

$$\inf_{(t,x)\in[0,1]\times\mathbb{R}^d} \sum_{i,j} v_i \frac{\partial^2 U_t(x)}{\partial x_i \partial x_j} v_j \geqslant K\|v\|^2.$$

Without loss of generality, the unique minimizer of U_0 is taken to be $x_0^\star = 0$.

(**A5**) (continuity in time)$\exists M < \infty$ such that

$$\|\nabla U_t(x) - \nabla U_s(x)\| \leqslant M|t-s|\sqrt{1+\|x - x^\star_{t\wedge s}\|^2}, \quad \forall x, t, s,$$

where $x_t^\star$ is the unique minimizer of U_t.

(**A6**) (bounded 3rd derivatives) The third order derivatives respect to x of $U_t(x)$ exist, are continuous, and bounded uniformly in t and x.

1.4 Discussion of the approach

For a review of methods for sampling from a log-concave distribution see [8, Sec. 7]. Notable recent contributions include [9], which gives bounds on the distance to stationarity in total variation, under a variety of assumptions on algorithm step size and the target density, including bounded perturbation of a log-concave density and strong log-concavity outside a ball. Under the latter assumption, convergence rates for Wasserstein distances and mean square error bounds for empirical averages of Lipschitz functions are given in [12]. Under conditions which allow for strong log concavity of the target distribution, exponential deviation inequalities of empirical averages of Lipschitz test functions are obtained in [21], and in the strongly log-concave case, mean square error bounds and exponential deviation inequalities for a discretized diffusion, again for Lipschitz tests functions, are obtained in the recent pre-print [10].

Compared to the assumptions in these works, which consider processes with a fixed invariant distribution, the time-uniform strong log-concavity assumption (A4) provides a natural starting point from which to analyze the time-inhomogeneous process $(X_t^\epsilon)_{t\in[0,1]}$. It seems likely that some of the techniques in the aforementioned works may be useful in helping relax this condition, but investigating this matter would lead to an even more lengthy and technical exposition. On the other hand, it should be noted that one of our key intermediate results, namely the commutation relation Lemma 18, cannot hold under anything weaker than (A4), see Remark 19, so one cannot expect results of precisely the same form as ours to hold more generally.

Lemma 18 allows us to establish Poincaré inequalities for the time-inhomogeneous process in section 3, which are amongst our main technical tools. A key reference for functional inequalities for inhomogeneous processes is [7], and some of our developments are informed by their approach. However we are not able to use their results directly since they do not accommodate our assumptions. In particular we explicitly work with possibly unbounded test functions $f_t(x)$ which may grow polynomially fast as $\|x\| \to \infty$, and this requires us to rigorously derive the results in section 3 from scratch.

The dimension dependence of errors for sequential Monte Carlo methods in discrete time was studied in [2, 3] under the assumption that the target distributions are of product form as in (5), and that the Markov transition kernels they consider factorize across dimensions in a similar manner. One of our main motivations is to relax that kind of independence assumption because it is unrealistic, although of course our setup is somewhat different to that of [2, 3], since we start from a continuous time perspective. It should also be noted that we do not consider any resampling operations.

The arXiv preprint [27] studies an algorithm for sampling from time-varying log-concave distributions. The process they work with is a discrete time Markov chain and conductance techniques are used in the analysis. Amongst their key assumptions are that the target distributions are supported on a compact convex subset of $\mathbb{R}^d$ and that one can compute an associated self-concordant barrier.

1.5 Statement of main results

Throughout section 1.5 and unless stated otherwise, ϵ is fixed to an arbitrary positive value, $(X_t^\epsilon)_{t\in[0,1]}$ is as in (1) with X_0^ϵ an $\mathcal{F}_0$-measurable random variable with distribution μ_0, and for $t \in (0,1]$, μ_t^ϵ is the distribution of X_t^ϵ.

1.5.1 Non-asymptotic variance and bias bounds

Theorem 1. *Fix $p \geqslant 1$, assume $\mu_0 \in \mathcal{P}^{2p}(\mathbb{R}^d)$ and that there exists a constant $K_0 > 0$ such that*

$$\operatorname{var}_{\mu_0}[f] \leqslant \frac{1}{K_0}\mu_0(\|\nabla f\|^2), \quad \forall f \in C_2^p(\mathbb{R}^d). \tag{13}$$

1) For each $t \in [0,1]$, the distribution μ_t^ϵ satisfies a Poincaré inequality:

$$\operatorname{var}_{\mu_t^\epsilon}[f] \leqslant \left[(1-e^{-Kt/\epsilon})\frac{1}{K} + e^{-Kt/\epsilon}\frac{1}{K_0}\right]\mu_t^\epsilon(\|\nabla f\|^2), \quad \forall f \in C_2^p(\mathbb{R}^d).$$

2) For any $f \in C_{0,2}^p([0,1]\times\mathbb{R}^d)$ such that $\pi_t f_t = 0$ for all $t \in [0,1]$, and any $h \in (0,1]$, define

$$S_\epsilon := \int_0^1 f_t(X_t^\epsilon)\mathrm{d}t, \qquad S_{\epsilon,h} := h\sum_{k=0}^{\lfloor 1/h\rfloor -1} f_{kh}(X_{kh}^\epsilon). \tag{14}$$

Then

$$\begin{aligned}
\operatorname{var}[S_\epsilon] &\leqslant \frac{2\epsilon}{K_0 \wedge K}\sup_{t\in[0,1]}\operatorname{var}_{\mu_t^\epsilon}[f_t],\\
|\mathbb{E}[S_\epsilon]| &\leqslant \frac{\epsilon}{K}\sup_{t\in[0,1]}\operatorname{var}_{\pi_t}[\phi_t]^{1/2}\sup_{t\in[0,1]}\operatorname{var}_{\pi_t}[f_t]^{1/2} + \alpha_p W^{(p)}(\mu_0,\pi_0)\frac{\epsilon}{K}\sup_{t\in[0,1]}\|\nabla f_t\|_p,
\end{aligned}$$

$$\begin{aligned}
\operatorname{var}[S_{\epsilon,h}] &\leqslant h\left(1+\frac{2}{1-e^{-(K_0\wedge K)h/\epsilon}}\right)\sup_{t\in[0,1]}\operatorname{var}_{\mu_t^\epsilon}[f_t],\\
|\mathbb{E}[S_{\epsilon,h}]| &\leqslant \frac{\epsilon}{K}\sup_{t\in[0,1]}\operatorname{var}_{\pi_t}[\phi_t]^{1/2}\sup_{t\in[0,1]}\operatorname{var}_{\pi_t}[f_t]^{1/2} + \frac{\alpha_p h}{1-e^{-Kh/\epsilon}}W^{(p)}(\mu_0,\pi_0)\sup_{t\in[0,1]}\|\nabla f_t\|_p,
\end{aligned}$$

where α_p, given in Lemma 14, is a constant depending only on ϵ, p, K, d, $\sup_{t\in(0,1)}\|\partial_t x_t^\star\|$, $\sup_{t\in[0,1]}\|x_t^\star\|$, and

$$W^{(p)}(\mu_0,\pi_0) := \inf_{\gamma\in\Gamma(\mu_0,\pi_0)}\int_{\mathbb{R}^{2d}}\left(1+\|x\|^{2p}\vee\|y\|^{2p}\right)\|x-y\|\gamma(\mathrm{d}x,\mathrm{d}y),$$

where $\Gamma(\mu_0,\pi_0)$ is the set of all couplings of μ_0 and π_0.

Proof. See section 2. □

Remark 2. See section 3.1.2 for discussion of the assumption in Theorem 1 that f is twice continuously differentiable w.r.t. x.

So far in section 1.5, the dimension d has been regarded as a constant. Our next task is to explicitly quantify the dependence on d of the variance and bias bounds in Theorem 1. We are particularly interested in growth which is at most polynomial in d. Pursuant to this, in the remainder of section 1.5.1 we adopt the perspective that d is an independent parameter on which various quantities may possibly depend, including h, ϵ and the quantities in hypothesis (A7) below, which we shall verify for an example in section 1.6. The phrasing of this hypothesis in terms of asymptotic behaviour as $d \to \infty$ is chosen for convenience, to achieve a balance between precision and ease of presentation in Corollary 3 of Theorem 1 below, its proof and application.

(**A7**) (Polynomial dependence on dimension) For a given $p \geqslant 1$, and for each $d \in \mathbb{N}$ a given $\mu_0 \in \mathcal{P}^{2p}(\mathbb{R}^d)$, K_0 satisfying (13), and $f \in C^p_{0,2}([0,1], \times\mathbb{R}^d)$, there exists a constant $q \geqslant 0$ independent of d such that, as $d \to \infty$,

$$W^{(p)}(\mu_0, \pi_0) \vee \sup_{t\in[0,1]} \|\nabla f_t\|_p \vee K^{-1} \vee K_0^{-1} \vee L^4 \vee \sup_{t\in[0,1]} \|x^\star_t\|^2 = O(d^q),$$

and

$$\mu_0(V^{2p}) = O(d^{q+1}).$$

Corollary 3. *Assume that the p, μ_0, K_0 and f in Theorem 1 satisfy (A7), and let q be as in the latter. If*

$$\frac{\epsilon}{K} \sup_{t\in(0,1)} \|\partial_t x^\star_t\| = O(1),$$

as $d \to \infty$, then

$$\mathrm{var}[S_\epsilon] = O\left(\frac{\epsilon}{K \wedge K_0} r_1(d)\right), \qquad |\mathbb{E}[S_\epsilon]| = O\left(\frac{\epsilon}{K} r_2(d) + \frac{\epsilon}{K} r_3(d)\right),$$

$$\mathrm{var}[S_{\epsilon,h}] = O\left(h\left[1 + \frac{2}{1 - e^{-(K_0\wedge K)h/\epsilon}}\right] r_1(d)\right), \qquad |\mathbb{E}[S_{\epsilon,h}]| = O\left(\frac{\epsilon}{K} r_2(d) + \frac{h}{1 - e^{-Kh/\epsilon}} r_3(d)\right),$$

where

$$r_1(d) := d^{4q+2p(q+1)+1}, \quad r_2(d) := d^{7q/4+3pq+3p/2+1/2}, \quad r_3(d) := d^{2q+pq+p}.$$

Proof. See section 2. ☐

1.5.2 A central limit theorem in the high-dimensional regime

The expressions in Corollary 3 suggest that the behaviour of $\mathrm{var}[S_{\epsilon,h}]$ and $|\mathbb{E}[S_{\epsilon,h}]|$ as $\epsilon \to 0$ depends on the scaling relationship between ϵ and h. We now introduce a parameter $\ell \geqslant 0$ to delineate two cases.

(**A8**) (ℓ-dependent scaling of h with ϵ)

1. In the case $\ell = 0$, we assume $h(\epsilon) = O(\epsilon^c)$ for an arbitrary $c > 1$.
2. In the case $\ell > 0$, we set $h(\epsilon) := \ell\epsilon$

Throughout the remainder of section 1.5.2, the value of $\ell \geqslant 0$ should be regarded as being chosen independently, and (A8) is assumed to hold.

To state our next main result we need to introduce some further notation. For each $s \in [0,1]$ and $\epsilon > 0$, let $(Y^{s,\epsilon}_t)_{t\in\mathbb{R}^+}$ be the solution of:

$$Y^{s,\epsilon}_t = Y^{s,\epsilon}_0 - \epsilon^{-1}\int_0^t \nabla U_s(Y^{u,\epsilon}_t)\mathrm{d}u + \sqrt{2\epsilon^{-1}}\int_0^t \mathrm{d}B_u,$$

where $Y^{s,\epsilon}_0$ is an $\mathcal{F}_0$-measurable random variable with distribution π_s. Then writing $L_2(\pi_s)$ for the collection of all real-valued functions that are square-integrable with respect to π_s, standard results for stationary reversible Markov processes and Markov chains ensure that for any $s \in [0,1]$ and $f_s \in L_2(\pi_s)$, the following limits exist:

$$\varsigma_0(s) := \lim_{\epsilon\to 0} \mathrm{var}\left[\epsilon^{-1/2}\int_0^1 f_s(Y^{s,\epsilon}_t)\mathrm{d}s\right],$$

$$\varsigma_\ell(s) := \lim_{\epsilon\to 0} \mathrm{var}\left[\epsilon^{-1/2}h(\epsilon)\sum_{k=0}^{\lfloor 1/h\rfloor-1} f_s(Y^{s,\epsilon}_{kh})\right], \quad \ell > 0.$$

With $Q^s_t(f)(y) := \mathbb{E}[f(Y^{s,1}_t)|Y^{s,1}_0 = y]$ and $\mathcal{L}_s f := -\langle\nabla U_s, \nabla f\rangle + \Delta f$, it is well known that the following bounds, in terms of $L_2(\pi_s)$ spectral gaps and constant K from (A4), hold:

$$\varsigma_\ell(s) \leqslant 2\mathrm{var}_{\pi_S}[f_s] \cdot \begin{cases} \mathrm{Gap}(\mathcal{L}_s)^{-1}, & \ell = 0, \\ \ell \mathrm{Gap}(Q_\ell^s)^{-1}, & \ell > 0, \end{cases}$$

and

$$\mathrm{Gap}(\mathcal{L}_s) \geqslant K, \qquad \mathrm{Gap}(Q_\ell^s)^{-1} \geqslant \frac{1-\exp(-K\ell)}{\ell}.$$

Indeed $\mathrm{Gap}(\mathcal{L}_s) \geqslant K$ is a direct consequence of the standard Poincaré inequality for the strongly log-concave distribution π_s. These bounds suggest that under hypotheses such as (A7), for each $s \in [0,1]$, fluctuations of the additive functionals $\int_0^1 f_s(Y_t^{s,\epsilon})\mathrm{d}s$ and $h(\epsilon)\sum_{k=0}^{\lfloor 1/h \rfloor -1} f_s(Y_{kh}^{s,\epsilon})$ associated with the time-homogeneous process $(Y_t^{s,\epsilon})_{t\in\mathbb{R}^+}$ could possibly be controlled by choosing ϵ^{-1} to be polynomial in d. Our next main result, Theorem 4, establishes that a similar phenomenon holds for additive functionals associated with time-inhomogeneous process $(X_t^\epsilon)_{t\in[0,1]}$.

Under our assumptions, for any $\ell \geqslant 0$, $s \mapsto \varsigma_\ell(s)$ can be shown to be integrable (see the proof of Lemma 53), and therefore

$$\sigma_\ell^2 := \int_0^1 \varsigma_\ell(s)\mathrm{d}s \tag{15}$$

is well-defined. In the context of Theorem 4 below, it is important to note that ς_ℓ and σ_ℓ^2 depend on the dimension d, but this dependence is not shown in the notation.

Theorem 4. *Fix $p \geqslant 1$ and for each $d \in \mathbb{N}$, fix a function $f \in C_{1,2}^p([0,1]\times\mathbb{R}^d)$ such that for each $t \in [0,1]$ $\pi_t f_t = 0$, and a probability measure $\mu_0 \in \mathcal{P}^{2p}(\mathbb{R}^d)$ and a constant $K_0 > 0$ satisfying (13). Assume that (A7) holds and assume addtionally that for each $\ell \geqslant 0$, $\sup_t 1/\varsigma_\ell(t)$ and $\sup_t \|\partial_t f_t\|_p$ grow at most polynomially fast as $d \to \infty$. Then for any $\ell \geqslant 0$ there exists $a > 0$ such that with $\epsilon(d) = O(d^{-a})$ and $d \mapsto h(d)$ such that (A8) holds,*

$$\lim_{d\to\infty} \left| \mathrm{var}\left[\epsilon(d)^{-1/2} S_{\epsilon(d),h(d)} - \sigma_\ell^2\right]\right| = 0,$$

and

$$\lim_{d\to\infty} \sup_{w\in\mathbb{R}} \left| \mathbb{P}\left[\epsilon(d)^{-1/2} S_{\epsilon(d),h(d)} / \sqrt{\sigma_\ell^2} \leqslant w\right] - \Phi(w)\right| = 0,$$

where $S_{\epsilon,h}$ is as in Theorem 1, and Φ is the standard Gaussian c.d.f.

Proof. See section 5. □

Remark 5. It is in principle possible to calculate quantitative bounds on the rates of convergence in Theorem 4, by agreggation of various bounds found in our proof. We do not pursue this here due to a lack of space and the limited interest of such bounds in practice.

Remark 6. Note that compared to Theorem 1, Theorem 4 requires additional assumptions that $s \mapsto f_s(x)$ is continuously differentiable for any $x \in \mathbb{R}^d$. This condition is required in order to obtain explicit control on the error in Rieman sums involved in our calculations, and could be relaxed easily to Hölder continuity, at the expense of additional notation.

Remark 7. As an aside, it is natural to investigate the impact of ℓ on the asymptotic variance σ_ℓ^2. Theorem 35 establishes that σ_ℓ^2 is a non-decreasing functions of ℓ. This result can be understood as being a generalisation of [16, Theorem 3.3], an important fact in the area of discrete time Markov chain Monte Carlo methods, concerned with "thinning" in the context of ergodic averages.

Remark 8. By inspecting the proofs in section 5, one can check that similar statements hold in the fixed dimension case, that is with $d \in \mathbb{N}$ held constant and $h(\epsilon)$ as in (A8),

$$\lim_{\epsilon\to 0} \left| \mathrm{var}\left[\epsilon^{-1/2} S_{\epsilon,h(\epsilon)} - \sigma_\ell^2\right]\right| = 0, \qquad \lim_{\epsilon\to\infty} \sup_{w\in\mathbb{R}} \left| \mathbb{P}\left[\epsilon^{-1/2} S_{\epsilon,h(\epsilon)} / \sqrt{\sigma_\ell^2} \leqslant w\right] - \Phi(w)\right| = 0.$$

1.5.3 Discretization of the process

One typically resorts to simulating some approximation to the diffusion $(X_t^\epsilon)_{t\in[0,1]}$ involving discretization in order to obtain a practical approximation to S_ϵ or $S_{\epsilon,h}$. There are many possible approaches to discretization of diffusions and it is not our objective to investigate or discuss their relative merits. Instead, we consider a simple Euler-Maruyama discretization scheme, since it is a generally applicable method whose practical computational cost is easy to assess and whose approximation properties can be quite directly analyzed.

We present next a general purpose lemma which allows control of moments of functions on the path space of one diffusion in terms of those of another, which we shall subsequently apply to the Euler-Maruyama discretization scheme.

Let E be the Polish space of continuous functions $z : t \in [0,1] \mapsto z_t \in \mathbb{R}^d$ endowed with the metric $\rho(z,\tilde{z}) = \sup_{t\in[0,1]} \|z_t - \tilde{z}_t\|$, and let $\mathcal{B}(E)$ be its Borel σ-algebra.

Lemma 9. *For any $(E,\mathcal{B}(E))$-valued random elements $X, \widetilde{X}$, any measurable function $\varphi : (E,\mathcal{B}(E)) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$, and any $p,q,r \in [1,+\infty)$ such that $1/q + 1/r = 1$,*

$$\sup_{c\in\mathbb{R}} \left|\mathbb{P}[\varphi(\widetilde{X}) \leqslant c] - \mathbb{P}[\varphi(X) \leqslant c]\right| \leqslant \|\mu - \tilde{\mu}\|_{\mathrm{tv}},$$

$$\mathbb{E}[|\varphi(\widetilde{X})|^p]^{1/p} \leqslant \mathbb{E}[|\varphi(X)|^p]^{1/p} + \|\mu - \tilde{\mu}\|_{\mathrm{tv}}^{1/pq} \left\{\mathbb{E}[|\varphi(X)|^{pr}]^{1/pr} + \mathbb{E}[|\varphi(\widetilde{X})|^{pr}]^{1/pr}\right\}.$$

where

$$\mu(A) := \mathbb{P}[X \in A], \qquad \tilde{\mu}(A) := \mathbb{P}[\widetilde{X} \in A], \quad A \in \mathcal{B}(E).$$

Proof. See section 2. □

For $\epsilon > 0$ and $h \in (0,1]$, let $\widetilde{X}^{\epsilon,h} = (\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$ be the solution of

$$\widetilde{X}_t^{\epsilon,h} = X_0^\epsilon - \epsilon^{-1}\int_0^t \widetilde{\nabla U}_s(\widetilde{X}_s^{\epsilon,h})\mathrm{d}s + \sqrt{2\epsilon^{-1}}\int_0^t \mathrm{d}B_s, \tag{16}$$

where X_0^ϵ is the same $\mathcal{F}_0$-measurable random variable with distribution μ_0 as in (1), and the following short-hand notation is used:

$$\widetilde{\nabla U}_t(\widetilde{X}_t^{\epsilon,h}) := \sum_{k=0}^{\lfloor 1/h\rfloor - 1} \nabla U_{kh}(\widetilde{X}_{kh}^{\epsilon,h})\mathbb{I}_{[kh,(k+1)h)}(t). \tag{17}$$

In practice, one does not simulate the entire trajectory $(\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$ but rather the skeleton $(\widetilde{X}_{kh}^{\epsilon,h})_{k=0,\ldots,\lfloor 1/h\rfloor-1}$. The point of writing (16)-(17) is to highlight that the term $\sqrt{2\epsilon^{-1}}\int_0^t \mathrm{d}B_s$ is common to both (16) and (1) so that the laws of $(X_t^\epsilon)_{t\in[0,1]}$ and $(\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$ are mutually absolutely continuous. Via Girsanov's theorem and Pinsker's inequality, Dalalyan [8] when studying a time-homogeneous process used this fact to estimate the total variation distance between the time-marginal distributions of a Langevin diffusion and its discretization, analogous in the present context to the distributions of say X_1^ϵ and $\widetilde{X}_1^{\epsilon,h}$. However, this Girsanov/Pinsker technique allows one to estimate the total variation distance not only between time-marginal distributions, but also between the laws of $(X_t^\epsilon)_{t\in[0,1]}$ and $(\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$, i.e. the probability measures

$$\mu^\epsilon(A) := \mathbb{P}[X^\epsilon \in A] \qquad \tilde{\mu}^{\epsilon,h}(A) := \mathbb{P}[\widetilde{X}^{\epsilon,h} \in A], \qquad A \in \mathcal{B}(E),$$

and we shall exploit that fact in the application of Lemma 9 in Section 1.6 to transfer the distributional convergence in Theorem 4 to the discretized process. In particular, Proposition 10 together with standard Foster-Lyapunov techniques will be applied to control the terms in the bounds of Lemma 9.

Proposition 10. *For any $q \geqslant 0$, if*

$$M^2 \vee L^4 \vee K^{-1} \vee \sup_t \|\partial_t x_t^\star\|^2 = O(d^q), \quad \mu_0(V) = O(d^{q+1}), \tag{18}$$

$$h \vee \epsilon \vee \frac{h}{\epsilon}\frac{L^2}{K} = o(1), \quad \frac{h}{\epsilon}d = O(1),$$

as $d \to \infty$, then

$$\|\mu^{\epsilon} - \tilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}} = O\left(\sqrt{\frac{h}{\epsilon^2} d^{4q+1}}\right).$$

Proof. See section 8.2. □

1.6 Example: Marginal likelihood computation for logistic regression

1.6.1 Model specification and verification of assumptions

Consider observations $Y_1, \dots, Y_m$ each valued in $\{0, 1\}$, covariate vectors $c_1, \dots, c_m$ each valued in $\mathbb{R}^d$, and an unknown parameter vector $x \in \mathbb{R}^d$. The observations are modelled as conditionally independent given the covariates and x, with the conditional probability of $\{Y_i = 1\}$ being $\varrho_i(x) := 1/(1 + e^{-\langle x, c_i \rangle})$. In a Bayesian approach to statistical inference we place an isotropic Gaussian prior distribution over the unknown parameter x, with covariance matrix $I_d/\tilde{\sigma}^2$. The posterior density over x has density on $\mathbb{R}^d$ proportional to:

$$\exp\left\{ y^T C x - \sum_{i=1}^{m} \log(1 + e^{\langle x, c_i \rangle}) - \frac{\|x\|^2}{2\tilde{\sigma}^2} \right\},$$

with the vector $y := (y_1, \dots, y_m)^T$ and matrix C whose ith row is c_i.

Let the functions $(U_t)_{t \in [0,1]}$ be given by

$$U_t(x) = -t y^T C x + t \sum_{i=1}^{m} \log(1 + e^{\langle x, c_i \rangle}) + \frac{\|x\|^2}{2\tilde{\sigma}^2}, \tag{19}$$

Then the distributions π_0 and π_1 specified by U_0 and U_1 are respectively the prior and posterior. Evaluating the "marginal likelihood" $Z_1 = \int_{\mathbb{R}^d} \exp\{-U_1(x)\} \mathrm{d}x$ allows one to assess the quality of model fit.

We shall now verify assumptions (A1)-(A6). We have

$$\nabla U_t(x) = -t y^T C + t \sum_{i=1}^{m} c_i \varrho_i(x) + \frac{x}{\tilde{\sigma}^2}, \qquad \nabla^{(2)} U_t(x) = t \sum_{i=1}^{m} \varrho_i(x)\{1 - \varrho_i(x)\} c_i c_i^T + \frac{I_d}{\tilde{\sigma}^2}. \tag{20}$$

$$\frac{\partial^3 U_t(x)}{\partial x_j \partial x_k \partial x_\ell} = t \sum_{i=1}^{m} c_{ij} c_{ik} c_{i\ell} \varrho_i(x)\{1 - \varrho_i(x)\}\{1 - 2\varrho_i(x)\} \tag{21}$$

where c_{ij} is the jth element of c_i.

By inspection of (19)-(20), (A1) holds with $p_0 = 1$. By considering the spectral norm of $\nabla^{(2)} U_t$, one obtains

$$\sup_{t \in [0,1]} \|\nabla U_t(x) - \nabla U_t(y)\| \leqslant (0.25 m \lambda + \tilde{\sigma}^{-2}) \|x - y\|,$$

where $\lambda_{\max}$ is the largest eigenvalue of $m^{-1} \sum_{i=1}^{m} c_i c_i^T$, and with

$$\xi := \|y^T C\| + \sum_{i=1}^{m} \|c_i\|, \tag{22}$$

we have

$$\|\nabla U_t(x)\| \leqslant (\xi \vee \tilde{\sigma}^{-2})(1 + \|x\|), \qquad \|\nabla U_t(x) - \nabla U_s(y)\| \leqslant \xi |t - s|.$$

So for the constants appearing in (A2)-(A5) one make take

$$K = \frac{1}{\tilde{\sigma}^2}, \quad L = \left(0.25 m \lambda_{\max} + \frac{1}{\tilde{\sigma}^2}\right) \vee \left(\xi \vee \frac{1}{\tilde{\sigma}^2}\right), \quad M = \xi. \tag{23}$$

(A6) is satisfied by inspection of (21).

1.6.2 Dimension dependence of the error

Let us now discuss application of Theorems 1 and 4. Observe from (19) that we have

$$\partial_t U_t(x) = -y^T Cx + \sum_{i=1}^{m} \log(1 + e^{\langle x, c_i \rangle}), \tag{24}$$

and define

$$\Delta_{\epsilon,h} := -h \sum_{k=0}^{\lfloor 1/h \rfloor - 1} \partial_t U_t(\widetilde{X}^{\epsilon,h}_{kh})\Big|_{t=kh} - \log \frac{Z_1}{Z_0},$$

where $(\widetilde{X}^{\epsilon,h}_t)_{t\in[0,1]}$ is as in (16).

Consider the following condition:

(**A9**) (Polynomial dependence on dimension for logistic regression) There exists $q \geqslant 0$ such that:

$$\tilde{\sigma}^2 \vee \left(0.25 m \lambda_{\max} + \frac{1}{\tilde{\sigma}^2}\right) \vee \xi = O(d^{q/4})$$

as $d \to \infty$.

In the proof of the following proposition, (A9) allows us to verify (A7), apply Corollary 3 and Theorem 4 with

$$f_t = -\partial_t U_t + \pi_t(\partial_t U_t), \tag{25}$$

and Proposition 10 and Lemma 9.

Proposition 11. *Assume that $\mu_0 = \pi_0$ and that (A9) holds for some given q.*
1) If

$$h \vee \epsilon = o(1), \qquad \frac{h}{\epsilon^2} d^{3q/2+1} \vee \epsilon d^{7q+3} = O(1) \tag{26}$$

as $d \to \infty$, then

$$\mathbb{E}[|\Delta_{\epsilon,h}|] = O\left(\sqrt{\epsilon d^{7q+3}} + \left[\frac{h}{\epsilon^2}\right]^{1/4} d^{9(q+1)/4} + h d^{5q+2}\right).$$

2) If

$$\left[\inf_{t\in[0,1]} t^2 \sum_{j=1}^{d} \left\{ \int_{\mathbb{R}^d} l(y;x) \left[\sum_{i=1}^{m} (y_i - \varrho_i(x)) c_{ij} - \frac{x_j}{\tilde{\sigma}^2}\right] \pi_t(\mathrm{d}x) \right\}^2 \right]^{-1} \tag{27}$$

grows at most polynomially fast as $d \to \infty$, where $l(y;x)$ is the log-likelihood:

$$l(y;x) := -y^T Cx + \sum_{i=1}^{m} \log(1 + e^{\langle x, c \rangle}),$$

then for any $c > 2$, there exists $a > 0$ such that with $\epsilon = O(d^{-a})$ and $h = \epsilon^c$,

$$\lim_{d\to\infty} \sup_{w\in\mathbb{R}} \left| \mathbb{P}\left[\epsilon(d)^{-1/2} \Delta_{\epsilon(d),h(d)} / \sqrt{\sigma_0^2} \leqslant w\right] - \Phi(w) \right| = 0,$$

where σ_0^2 is as in (15) with f_t as in (25).

Proof. See section 2. □

2 Proofs for section 1

Proof of Theorem 1. Write $P_{0,t}f(x) = \mathbb{E}[f(X_t^\epsilon)|X_0 = x]$ so that $\mu_t^\epsilon = \mu_0 P_{0,t}$. For part 1) note that by Lemma 22 applied with $\nu = \mu_0$,

$$\begin{aligned}
\mathrm{var}_{\mu_t^\epsilon}[f] &\leqslant \left[(1-e^{-2Kt/\epsilon})\frac{1}{K} + e^{-2Kt/\epsilon}\frac{1}{K_0}\right]\mu_t^\epsilon(\|\nabla f\|^2) \\
&\leqslant \frac{1}{K_0 \wedge K}\mu_t^\epsilon(\|\nabla f\|^2), \qquad \forall f \in C_2^p(\mathbb{R}^d),
\end{aligned}$$

and then by Cauchy-Schwartz and Lemma 23 applied with $\kappa_\nu(u) = K_0 \wedge K$,

$$\begin{aligned}
|\mathbb{E}[(f_s(X_s^\epsilon) - \mu_s^\epsilon f_s)(f_t(X_t^\epsilon) - \mu_t^\epsilon f_t)]| &\leqslant \mathrm{var}_{\mu_s^\epsilon}[f_s]^{1/2}\mathrm{var}_{\mu_s^\epsilon}[P_{s,t}f_t]^{1/2} \\
&\leqslant \mathrm{var}_{\mu_s^\epsilon}[f_s]^{1/2}\mathrm{var}_{\mu_t^\epsilon}[f_t]^{1/2}e^{-(K_0\wedge K)(t-s)/\epsilon}.
\end{aligned}$$

Therefore, for part 2),

$$\begin{aligned}
\mathrm{var}[S_\epsilon] &= \mathbb{E}\left[\left(\int_0^1 f_t(X_t^\epsilon) - \mu_t^\epsilon f_t \mathrm{d}t\right)^2\right] \\
&= 2\mathbb{E}\left[\int_0^1\int_s^1 (f_s(X_s^\epsilon) - \mu_s^\epsilon f_s)(f_t(X_t^\epsilon) - \mu_t^\epsilon f_t)\mathrm{d}t\mathrm{d}s\right] \\
&\leqslant 2\sup_t \mathrm{var}_{\mu_t^\epsilon}[f_t]\int_0^1\int_s^1 e^{-(K_0\wedge K)(t-s)/\epsilon}\mathrm{d}t\mathrm{d}s \\
&\leqslant 2\frac{\epsilon}{K_0\wedge K}\sup_t \mathrm{var}_{\mu_t^\epsilon}[f_t].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{var}[S_{\epsilon,h}] &= \mathbb{E}\left[\left(h\sum_{k=0}^{\lfloor 1/h\rfloor -1} f_{kh}(X_{kh}^\epsilon) - \mu_{kh}^\epsilon f_{kh}\right)^2\right] \\
&\leqslant h^2\sum_{k=0}^{\lfloor 1/h\rfloor -1}\mathrm{var}_{\mu_{kh}^\epsilon}[f_{kh}] + 2h^2\sum_{k=0}^{\lfloor 1/h\rfloor -1}\sum_{j>k}\mathrm{var}_{\mu_{kh}^\epsilon}[f_{kh}]^{1/2}\mathrm{var}_{\mu_{kh}^\epsilon}[P_{kh,jh}f_{jh}]^{1/2} \\
&\leqslant \left(h + 2h^2\sum_{k=0}^{\lfloor 1/h\rfloor -1}\sum_{j>k}e^{-(K_0\wedge K)(j-k)h/\epsilon}\right)\sup_t \mathrm{var}_{\mu_t^\epsilon}[f_t] \\
&\leqslant h\left(1 + \frac{2}{1-e^{-(K_0\wedge K)h/\epsilon}}\right)\sup_t \mathrm{var}_{\mu_t^\epsilon}[f_t].
\end{aligned}$$

For the bias bounds, we have by Lemmas 26 and 24,

$$\begin{aligned}
|\mathbb{E}[f_t(X_t^\epsilon)]| &= |\mu_0 P_{0,t}f_t| \\
&\leqslant |\pi_0 P_{0,t}f_t - \pi_t f_t| + |(\mu_0 - \pi_0)P_{0,t}f_t| \\
&\leqslant \sup_{s\in[0,t]}\mathrm{var}_{\pi_s}[\phi_s]^{1/2}\mathrm{var}_{\pi_t}[f_t]^{1/2}\frac{\epsilon}{K}(1-e^{-Kt/\epsilon}) \\
&\quad + \alpha_p\|\nabla f_t\|_p W^{(p)}(\mu_0,\pi_0)e^{-Kt/\epsilon}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mathbb{E}[S_\epsilon]| &\leqslant \sup_t \mathrm{var}_{\pi_t}[\phi_t]^{1/2}\sup_t \mathrm{var}_{\pi_t}[f_t]^{1/2}\frac{\epsilon}{K} + \alpha_p W^{(p)}(\mu_0,\pi_0)\int_0^t e^{-Kt/\epsilon}\mathrm{d}t \sup_t\|\nabla f_t\|_p \\
&= \sup_t \mathrm{var}_{\pi_t}[\phi_t]^{1/2}\mathrm{var}_{\pi_t}[f_t]^{1/2}\frac{\epsilon}{K} + \alpha_p W^{(p)}(\mu_0,\pi_0)\frac{\epsilon}{K}\sup_t\|\nabla f_t\|_p,
\end{aligned}$$

and

$$
\begin{aligned}
|\mathbb{E}[S_{\epsilon,h}]| &\leqslant h \sum_{k=0}^{\lfloor 1/h\rfloor -1} |\mathbb{E}[f_{kh}(X_{kh}^{\epsilon})]| \\
&\leqslant \sup_t \mathrm{var}_{\pi_t}[\phi_t]^{1/2} \sup_t \mathrm{var}_{\pi_t}[f_t]^{1/2} \frac{\epsilon}{K} h \sum_{k=0}^{\lfloor 1/h\rfloor -1} \left[1 - e^{-khK/\epsilon}\right] \\
&+ \alpha_p \sup_t \|\nabla f_t\|_p h W^{(p)}(\mu_0, \pi_0) \sum_{k=0}^{\lfloor 1/h\rfloor -1} e^{-Kkh/\epsilon} \\
&\leqslant \sup_t \mathrm{var}_{\pi_t}[\phi_t]^{1/2} \sup_t \mathrm{var}_{\pi_t}[f_t]^{1/2} \frac{\epsilon}{K} + \sup_t \|\nabla f_t\|_p \frac{\alpha_p h}{1 - e^{-hK/\epsilon}} W^{(p)}(\mu_0, \pi_0).
\end{aligned}
$$

□

Proof of Corollary 3. Let us first obtain upper bounds on:

$$
\sup_t \mathrm{var}_{\mu_t^{\epsilon}}[f_t], \quad \sup_t \mathrm{var}_{\pi_t}[\phi_t] \quad \sup_t \mathrm{var}_{\pi_t}[f_t],
$$

By part 1) of Theorem 1, Lemma 14, Lemma 70 and (A7),

$$
\begin{aligned}
\sup_t \mathrm{var}_{\mu_t^{\epsilon}}[f_t] &\leqslant \sup_t \frac{\mu_t^{\epsilon}(\|\nabla f_t\|^2)}{K \wedge K_0} \\
&\leqslant \frac{3}{K \wedge K_0} \sup_t \mu_t^{\epsilon}(\bar{V}^{(2p)}) \sup_{t\in[0,1]} \|\nabla f_t\|_p^2 \\
&\leqslant \frac{3\alpha_{2p}}{K \wedge K_0} [1 + \mu_0(V^{2p})] \sup_t \|\nabla f_t\|_p^2 \\
&= O(d^q d^{2p(q+1)} d^{q+1} d^{2q}) \\
&= O(d^{4q+2p(q+1)+1}).
\end{aligned}
$$

By Remark 21, (A3), Lemma 71 with there $p=1$, and (A7),

$$
\sup_t \mathrm{var}_{\pi_t}[\phi_t] \leqslant \frac{1}{K} \sup_t \pi_t(\|\nabla U_t\|^2) \leqslant \frac{3L^2}{K} \sup_t \pi_t(\bar{V}) = O(d^{3q/2} d^{q+1}) = O(d^{5q/2+1}).
$$

Lastly, $\sup_t \mathrm{var}_{\pi_t}[f_t]$ can be similarly controlled using Remark 21, (A7) and Lemma 71, to give

$$
\sup_{t\in[0,1]} \mathrm{var}_{\pi_t}[f_t] \leqslant \frac{1}{K} \sup_{t\in[0,1]} \pi_t(\bar{V}^{2p}) \sup_{t\in[0,1]} \|\nabla f_t\|_p^2 = O(d^q d^p d^{2pq+2p}) = O(d^{q+p(3+2q)}).
$$

Using the above estimates, we have from the expressions in Theorem 1 and Lemma 70,

$$
\begin{aligned}
\mathrm{var}[S_\epsilon] &= O\left(\frac{\epsilon}{K_0 \wedge K} d^{4q+2p(q+1)+1}\right), \\
|\mathbb{E}[S_\epsilon]| &= O\left(\frac{\epsilon}{K} d^{5q/4+1/2} d^{q/2+p(3+2q)/2} + d^{p(q+1)} d^q \frac{\epsilon}{K} d^q\right) \\
&= O\left(\frac{\epsilon}{K} d^{7q/4+3pq+3p/2+1/2} + \frac{\epsilon}{K} d^{2q+pq+p}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{var}[S_{\epsilon,h}] &= O\left(h\left(1 + \frac{2}{1 - e^{-(K_0\wedge K)h/\epsilon}}\right) d^{4q+2p(q+1)+1}\right) \\
|\mathbb{E}[S_{\epsilon,h}]| &= O\left(\frac{\epsilon}{K} d^{7q/4+3pq+3p/2+1/2} + \frac{h}{1 - e^{-Kh/\epsilon}} d^{2q+pq+p}\right).
\end{aligned}
$$

□

Proof of Lemma 9. The first inequality is an immediate consequence of the definition of the total variation distance. For the second inequality, since E is Polish there exists a maximal coupling of $X, \widetilde{X}$, [26, Ch. I, Sec. 5, p. 18], that is a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \mathbf{P})$ on which are defined two $(E, \mathcal{B}(E))$-valued random elements $Z, \widetilde{Z}$ such that

$$\mathbf{P}[Z \in A] = \mu(A), \quad \mathbf{P}[\widetilde{Z} \in A] = \widetilde{\mu}(A), \quad A \in \mathcal{B}(E),$$
$$\mathbf{P}[Z \neq \widetilde{Z}] = \|\mu - \widetilde{\mu}\|_{\mathrm{tv}}.$$

With expectation w.r.t. $\mathbf{P}$ denoted by $\mathbf{E}$, we then have, using Holder's inequality,

$$\begin{aligned}
\mathbb{E}[|\varphi(\widetilde{X})|^p]^{1/p} &= \mathbf{E}[|\varphi(\widetilde{Z})|^p]^{1/p} \\
&\leqslant \mathbf{E}[|\varphi(Z)|^p]^{1/p} + \mathbf{E}[|\varphi(\widetilde{Z}) - \varphi(Z)|^p]^{1/p} \\
&= \mathbb{E}[|\varphi(X)|^p]^{1/p} + \mathbf{E}[\mathbb{I}\{Z \neq \widetilde{Z}\}|\varphi(\widetilde{Z}) - \varphi(Z)|^p]^{1/p} \\
&\leqslant \mathbb{E}[|\varphi(X)|^p]^{1/p} + \mathbf{P}[Z \neq \widetilde{Z}]^{1/pq}\mathbf{E}[|\varphi(\widetilde{Z}) - \varphi(Z)|^{pr}]^{1/pr} \\
&\leqslant \mathbb{E}[|\varphi(X)|^p]^{1/p} + \|\mu - \widetilde{\mu}\|_{\mathrm{tv}}^{1/pq}\left\{\mathbb{E}[|\varphi(X)|^{pr}]^{1/pr} + \mathbb{E}[|\varphi(\widetilde{X})|^{pr}]^{1/pr}\right\}.
\end{aligned}$$

□

Lemma 12. *If (A9) holds for some given q, then f_t taken to be*

$$f_t(x) = -\partial_t U_t(x) + \pi_t(\partial_t U_t), \tag{28}$$

and K, L, M as in (23) satisfy

$$\sup_{t\in[0,1]} \|\nabla f_t\|_1 \vee K^{-1} \vee L^4 \vee M^2 \vee \sup_t \|x_t^\star\|^2 \vee \sup_t \|\partial_t x_t^\star\|^2 = O(d^q),$$

and π_0 as in (2) with U_0 as in (19) satisfies

$$\pi_0(V) = O(d^{q+1}),$$

as $d \to \infty$.

Proof. By Lemma 68, (23) and (A9),

$$\sup_t \|\partial_t x_t^\star\| \vee \sup_t \|x_t^\star\| \leqslant \frac{M}{K} = \xi\tilde{\sigma}^2 = O(d^{q/2}).$$

This fact together with $K^{-1} = \tilde{\sigma}^2 = O(d^{q/4})$ by (A9) validates an application of Lemma 71 with there $p = 1$ to give

$$\pi_0(V) = O(d^{q+1}).$$

Once more using (A9),

$$\sup_t \|\nabla f_t\|_1 \leqslant \|y^T C\| + \sum_{i=1}^m \|c_i\| = \xi = O(d^{q/4}).$$

The proof is complete since (A9) directly implies that $L^4 = (0.25m\lambda_{\max} + \tilde{\sigma}^{-2})^4 \vee (\xi \vee \tilde{\sigma}^{-2})^4 = O(d^q)$ and $M = \xi = O(d^{q/4})$. □

Proof of Proposition 11. For part 1), using Lemma 66 and Lemma 9, we have

$$\mathbb{E}[|\Delta_{\epsilon,h}|] \leqslant T_1(\epsilon, h) + T_2(h),$$

where

$$T_1(\epsilon, h) := \mathbb{E}\left[|S_{\epsilon,h}|\right] + \|\mu^\epsilon - \widetilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}}^{1/2}\left\{\mathbb{E}[|S_{\epsilon,h}|^2]^{1/2} + \mathbb{E}[|\widetilde{S}_{\epsilon,h}|^2]^{1/2}\right\}, \tag{29}$$

$$T_2(h) := \left| h\sum_{k=0}^{\lfloor 1/h\rfloor - 1} \pi_{kh}\left(\partial_t U_t\big|_{t=kh}\right) - \int_0^1 \pi_t(\partial_t U_t)\mathrm{d}t \right|, \tag{30}$$

$S_{\epsilon,h}$ is as in (14) with (28), and $\widetilde{S}_{\epsilon,h}$ is defined by replacing X^{ϵ}_{kh} in $S_{\epsilon,h}$ with $\widetilde{X}^{\epsilon}_{kh}$.

We shall estimate $T_1(\epsilon,h)$ using Corollary 3. To this end, note that Lemma 12 implies that (A7) is satisfied with there $p=1$; $\mu_0=\pi_0$ hence $K_0=K$, see Remark 21; and f as in (28). Also by Lemma 12, $K^{-1}=O(d^q)$ and $\sup_t \|\partial_t x^\star_t\| = O(d^{q/2})$, so the hypothesis of the proposition $\epsilon d^{7q+3}=O(1)$ implies $\epsilon \sup_t \|\partial_t x^\star_t\|/K = O(1)$. Therefore the hypotheses of Corollary 3 are satisfied, giving:

$$\begin{aligned}\mathbb{E}\left[|S_{\epsilon,h}|\right]^2 &\leqslant \mathbb{E}\left[|S_{\epsilon,h}|^2\right] = \mathrm{var}[S_{\epsilon,h}] + \mathbb{E}[S_{\epsilon,h}]^2 \\ &= O\left(h\left[1+\frac{2}{1-e^{-Kh/\epsilon}}\right]r_1(d) + \left[\frac{\epsilon}{K}r_2(d) + \frac{h}{1-e^{-Kh/\epsilon}}r_3(d)\right]^2\right),\end{aligned} \tag{31}$$

where

$$r_1(d)=d^{6q+3},\quad r_2(d)=d^{19q/4+2},\quad r_3(d)=d^{3q+1}. \tag{32}$$

Now (A9) implies that $K=\tilde{\sigma}^{-2}=O(d^{q/4})$, which combined with the hypotheses of the proposition $\epsilon = o(1)$ and $\frac{h}{\epsilon^2}d^{3q/2+1}=O(1)$ implies $Kh/\epsilon = o(1)$. Using this and the facts that by Lemma 12, $K^{-1}=O(d^q)$, and that the hypothesis of the proposition $\epsilon d^{7q+3}=O(1)$ implies $\epsilon d^{9q/2+1}=O(1)$, it follows from (31) and (32) that

$$\begin{aligned}\mathbb{E}\left[|S_{\epsilon,h}|\right] \leqslant \mathbb{E}[|S_{\epsilon,h}|^2]^{1/2} &= O\left(\sqrt{\left[h+\frac{\epsilon}{K}\right]r_1(d) + \left[\frac{\epsilon}{K}\{r_2(d)\vee r_3(d)\}\right]^2}\right) \\ &= O\left(\sqrt{\frac{\epsilon}{K}r_1(d) + \left[\frac{\epsilon}{K}r_2(d)\right]^2}\right) \\ &= O\left(\sqrt{\epsilon d^{7q+3}+\epsilon^2 d^{23q/2+4}}\right) \\ &= O\left(\sqrt{\epsilon d^{7q+3}(1+\epsilon d^{9q/2+1})}\right) \\ &= O\left(\sqrt{\epsilon d^{7q+3}}\right).\end{aligned}$$

For the second term in $T_1(\epsilon,h)$, first note that by Lemma 12, $L^2/K=O(d^{3q/2})$, which combined with the hypotheses of the proposition $\epsilon=o(1)$ and $\frac{h}{\epsilon^2}d^{3q/2+1}=O(1)$ implies $\frac{hL^2}{\epsilon K}=o(1)$ and $hd/\epsilon = O(1)$. These facts combined with Lemma 12 validate an application of Proposition 10 to give

$$\|\mu^\epsilon - \tilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}}^{1/2} = O\left(\left[\frac{h}{\epsilon^2}d^{4q+1}\right]^{1/4}\right). \tag{33}$$

Lemma 12 and (26) also validate an application of Lemma 65 to give

$$\mathbb{E}[|\widetilde{S}_{\epsilon,h}|^2]^{1/2} \leqslant \sup_t \|f_t^2\|_1^{1/2} h \sum_{k=0}^{\lceil 1/h\rceil -1}(1+\mathbb{E}[\|\widetilde{X}^{\epsilon,h}_{kh}\|^2]) = O\left(\sup_{t\in[0,1]}\|f_t^2\|_1^{1/2}\{\epsilon d^{2q+1}+hd^{q+1}+d^q\}\right), \tag{34}$$

where

$$\begin{aligned}\|f_t^2\|_1^{1/2} &\leqslant \sqrt{3\sup_x \frac{f_t(x)^2}{(1+\|x\|)^2}} = \sqrt{3}\|f_t\|_{1/2} \\ &\leqslant \sqrt{3}\left\{\sup_x \frac{|\partial_t U_t(x)|}{1+\|x\|} + \pi_t(\bar{V})\|\partial_t U_t\|_1\right\},\end{aligned} \tag{35}$$

$$
\begin{aligned}
\|\partial_t U_t\|_1 &\leqslant 3 \sup_x \frac{|\partial_t U_t(x)|}{1+\|x\|} \\
&\leqslant 3\|y^T C\| + 3\sum_{i=1}^d \sup_x \frac{\log(1+e^{\|x\|\|c_i\|})}{1+\|x\|} \\
&\leqslant 3\|y^T C\| + 3\sum_{i=1}^d \sup_x \frac{\log 2+\|x\|\|c_i\|}{1+\|x\|} \\
&\leqslant 3\|y^T C\| + 3d\log 2 + 3\sum_{i=1}^d \|c_i\| \\
&= O\left(d+\xi\right), \qquad (36)
\end{aligned}
$$

and by Lemma 71,

$$
\sup_{t\in[0,1]} \pi_t(\bar{V}) = O(d^{q+1}). \qquad (37)
$$

Combining (31)-(37) and using the hypotheses of the proposition $\epsilon = o(1)$ and $h = o(1)$, we find

$$
\begin{aligned}
\mathbb{E}[|\widetilde{S}_{\epsilon,h}|^2]^{1/2} &= O\left(\{(d+\xi)d^{q+1}\}\{\epsilon d^{2q+1} + h d^{q+1} + d^q\}\right) \\
&= O\left(d^q\{d(d+\xi) + \epsilon d^{q+1} + hd + 1\}\right) \\
&= O\left(d^{q+2} + d^{q+1}\xi\right).
\end{aligned}
$$

Collecting the above estimates for $\mathbb{E}\left[|S_{\epsilon,h}|\right]$, $\|\mu^\epsilon - \widetilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}}^{1/2}$, $\mathbb{E}[|S_{\epsilon,h}|^2]^{1/2}$, $\mathbb{E}[|\widetilde{S}_{\epsilon,h}|^2]^{1/2}$, returning to (29) and using that $\xi = O(d^{q/4})$ by (A9) and the hypothesis of the proposition $\epsilon d^{7q+3} = O(1)$, we have established

$$
\begin{aligned}
T_1(\epsilon,h) &= O\left(\sqrt{\epsilon d^{7q+3}} + \left[\frac{h}{\epsilon^2} d^{4q+1}\right]^{1/4}\left[\sqrt{\epsilon d^{7q+3}} + d^{q+2} + d^{q+1}\xi\right]\right) \\
&= O\left(\sqrt{\epsilon d^{7q+3}} + \left[\frac{h}{\epsilon^2} d^{4q+1}\right]^{1/4}\left[d^{q+2} + d^{5q/4+1}\right]\right) \\
&= O\left(\sqrt{\epsilon d^{7q+3}} + \left[\frac{h}{\epsilon^2}\right]^{1/4} d^{9(q+1)/4}\right).
\end{aligned}
$$

To estimate $T_2(h)$, an application of Lemma 63 with there $p=1$, $f_t = -\partial_t U_t$, $\beta = 1$, $R_f = 1$, $C_f = M = \xi$ as in (22) and $K = \tilde{\sigma}^{-2}$ as in (23), followed by Lemma 12 and Lemma 63, gives:

$$
\begin{aligned}
T_2(h) = \left| h \sum_{k=0}^{\lfloor 1/h\rfloor -1} \pi_{kh}\left(\partial_t U_t|_{t=kh}\right) - \int_0^1 \pi_t(\partial_t U_t)\mathrm{d}t \right| &\leqslant h^\beta \tilde{\alpha}_1\left(M \vee \sup_t \|\nabla \partial_t U_t\|_1\right)\left[1 + \tilde{\alpha}_1 \frac{M}{K} \sup_{t\in[0,1]} \sqrt{\bar{V}(x_t^\star)}\right] \\
&= O\left(hd^{2q+1}\left[1 + d^{5q/2+1}\sqrt{1+d^q}\right]\right) \\
&= O\left(hd^{5q+2}\right). \qquad (38)
\end{aligned}
$$

For part 2), first regard ϵ and h as fixed. Noting

$$
\Delta_{\epsilon,h} = \widetilde{S}_{\epsilon,h} - \left[h \sum_{k=0}^{\lfloor 1/h\rfloor -1} \pi_{kh}\left(\partial_t U_t|_{t=kh}\right) - \int_0^1 \pi_t(\partial_t U_t)\mathrm{d}t\right],
$$

and using the fact, established in the proof of Proposition 38, that for any two random variables Z_1 and Z_2 and any $\delta > 0$,

$$
\sup_{w\in\mathbb{R}} |\mathbb{P}[Z_1 + Z_2 \leqslant w] - \Phi(w)| \leqslant \sup_{w\in\mathbb{R}} |\mathbb{P}[Z_1 \leqslant w] - \Phi(w)| + \mathbb{P}[|Z_2| > \delta] + (2\pi)^{-1/2}\delta,
$$

we have

$$\sup_{w\in\mathbb{R}}\left|\mathbb{P}\left[\epsilon^{-1/2}\Delta_{\epsilon,h}/\sqrt{\sigma_0^2}\leqslant w\right]-\Phi(w)\right|\leqslant\sup_{w\in\mathbb{R}}\left|\mathbb{P}\left[\epsilon^{-1/2}S_{\epsilon,h}/\sqrt{\sigma_0^2}\leqslant w\right]-\Phi(w)\right| \tag{39}$$

$$+\sup_{w\in\mathbb{R}}\left|\mathbb{P}\left[\epsilon^{-1/2}S_{\epsilon,h}/\sqrt{\sigma_0^2}\leqslant w\right]-\mathbb{P}\left[\epsilon^{1/2}\widetilde{S}_{\epsilon,h}/\sqrt{\sigma_0^2}\leqslant w\right]\right| \tag{40}$$

$$+\,\mathbb{I}[\epsilon^{-1/2}|T_2(h)|/\sqrt{\sigma_0^2}>\delta]+(2\pi)^{-1/2}\delta. \tag{41}$$

Now let $\epsilon(d)$ and $h(d)$ be dependent on d as in the statement of part 2) of the proposition. Note that this places us in the case $\ell=0$ in (A8).

To show that the term on the right of the inequality in (39) converges to zero as $d\to\infty$, let us check the hypotheses of Theorem 4 in the case $\ell=0$. We have already established that (A7) is satisfied with there $p=1$, so it remains to check that $\sup_t\|\partial_t f_t\|_1$ and $\sup_t 1/\varsigma_0(t)$ grow at most polynomially fast as $d\to\infty$, where f_t is as in (24).

For $\sup_t\|\partial_t f_t\|_1$, note that f_t as in (24) does not depend on t and it is straightforward to check that $\partial_t f_t(x)=-\mathrm{var}_{\pi_t}[\partial_t U_t]$ for all x, so $\sup_t\|\partial_t f_t\|_1\leqslant\sup_t\pi_t[(\partial_t U_t)^2]\leqslant\sup_t\pi_t(\bar{V}^2)\|\partial_t U_t\|_1^2$, which grows at most polynormially fast as $d\to\infty$ by Lemma 71 and (36).

For $\sup_t 1/\varsigma_0(t)$, let us verify the hypotheses of Lemma 73 hold, i.e. that $\sup_s\|\tilde{\mathcal{L}}_s f_s\|_{p+1/2}$ and $\sup_{t\in[0,1]}1/\mathrm{var}_{\pi_t}[f_t]$ grow at most polynomially fast as $d\to\infty$. For the former, we have $|\tilde{\mathcal{L}}_s f_s|\leqslant\|\nabla U_s\|\|\nabla f_s\|+|\Delta f_s|$, and by (A3) and Lemma 12, $\|\nabla U_s\|_{1/2}\leqslant L=O(d^{q/4})$; also by Lemma 12, $\sup_s\|\nabla f_s\|_1=O(d^q)$, and $\frac{\partial^2 f_t}{\partial x_j^2}=-\sum_{i=1}^m c_{ij}^2\varrho_i(x)[1-\varrho_i(x)]$, hence $|\Delta f_t|\leqslant\sum_{i=1}^m\|c_i\|^2\leqslant\xi=O(d^{q/4})$ by (A9). Therefore indeed $\sup_s\|\tilde{\mathcal{L}}_s f_s\|_{p+1/2}$ grows at most polynomially fast as $d\to\infty$. By Lemma 69, $\mathrm{var}_{\pi_t}[f_t]\geqslant L^{-1}\sum_{i=1}^d\pi_t\left(\partial_t U_t\frac{\partial U_t}{\partial x_i}\right)^2$, and

$$-\pi_t\left(\partial_t U_t\frac{\partial U_t}{\partial x_j}\right)=t\int_{\mathbb{R}^d}l(y;x)\left(\sum_{i=1}^m c_{ij}\left(y_i-\varrho_i(x)\right)-\frac{x_j}{\tilde{\sigma}^2}\right)\mathrm{d}x,$$

so that under the hypothesis of the proposition that (27) grows no faster than polynomially, we have by Lemma 73 that $\sup_t 1/\varsigma_0(t)$ grows no faster than polynomially. Hence the term on the right of the inequality in (39) indeed converges to zero as $d\to\infty$.

By Lemma 9, Lemma 12 and Proposition 10, the term in (40) converges to zero as $d\to\infty$ thanks to the assumed scaling $h=\epsilon^c$ for some $c>2$ and $\epsilon=O(d^{-a})$ for $a>0$ large enough.

By (38), $\epsilon^{-1/2}|T_2(h)|=O(\epsilon^{-1/2}hd^{5q+2})$ and we have already established that $\sup_t 1/\varsigma_0(t)$ grows at most polynomially fast with d, hence the same is true of $1/\sqrt{\sigma_0^2}$. Therefore increasing a in $\epsilon=O(d^{-a})$ if necessary, and then choosing δ in (41) to go to zero suitably slowly as $d\to\infty$, the two terms in (41) tend to zero as $d\to\infty$.

We have shown that all the terms on the right of the inequality in (39)-(41) converge to zero as $d\to\infty$, and that completes the proof of the proposition.

$\square$

3 Poincaré inequalities, variance and bias decay for the inhomogeneous Langevin diffusion

Throughout section 3, $\epsilon > 0$ is a fixed constant.

3.1 Preliminaries about the process

3.1.1 Existence and Lipschitz continuity with respect to initial conditions

Let $(B_t)_{t\in[0,1]}$ be d-dimensional Brownian motion. Under (A2), (A3) and (A5), for each $s \in [0,1]$ there exists a strong solution to:

$$X_{s,t}^x = x - \epsilon^{-1}\int_s^t \nabla U_u(X_{s,u}^x)\mathrm{d}u + \sqrt{2\epsilon^{-1}}\int_s^t \mathrm{d}B_u, \quad t \in [s,1]. \tag{42}$$

pathwise uniqueness holds, see for example [11, Thm. 2.9, p.190], [23, Thm 3.4, p. 71] or [17, Thm. 4, p. 402], and the solution is non-explosive [23, p. 75]. Moreover, as noted by [24, Thm. 2.2, Ch. 2, p. 211], we can work with a version of $X_{s,t}^x$ which is continuous in s,t,x almost surely, and satisfies (42) for all s,t,x, almost surely.

Throughout section 3, we take:

$$P_{s,t}f(x) := \mathbb{E}[f(X_{s,t}^x)], \qquad \mathcal{L}_t f := -\epsilon^{-1}\langle \nabla U_t, \nabla f\rangle + \epsilon^{-1}\Delta f,$$

with the dependence on ϵ suppressed from the notation.

We shall make extensive use of the following observation, noted in the time-homogeneous case by [5].

Lemma 13. *Almost surely, the following holds for all x,y and $s \leqslant t$,*

$$\|X_{s,t}^x - X_{s,t}^y\| \leqslant e^{-K(t-s)/\epsilon}\|x-y\|.$$

Proof. Ito's lemma gives

$$\begin{aligned}
&e^{2K(t-s)/\epsilon}\|X_{s,t}^x - X_{s,t}^y\|^2\\
&= \|x-y\|^2\\
&+\frac{2}{\epsilon}\int_s^t \left(K\|X_{s,u}^x - X_{s,u}^y\|^2 - \left\langle \nabla U_u(X_{s,u}^x) - \nabla U_u(X_{s,u}^y)\,,\, X_{s,u}^x - X_{s,u}^y\right\rangle\right) e^{2K(u-s)/\epsilon}\mathrm{d}u,
\end{aligned}$$

and by Lemma 67, (A4) is equivalent to

$$\langle \nabla U_t(x) - \nabla U_t(y), x-y\rangle \geqslant K\|x-y\|^2, \quad \forall x,y.$$

□

3.1.2 Drift, regularity and validity of forward and backward equations

Lemma 14. *For any $p \geqslant 1$ and $\kappa \in (0, Kp)$ define:*

$$\begin{aligned}
\delta &:= \epsilon^{-1}(Kp-\kappa),\\
r &:= \frac{p}{\kappa}\epsilon \sup_{t\in(0,1)}\|\partial_t x_t^\star\| + \sqrt{\frac{p^2}{\kappa^2}\epsilon^2 \sup_{t\in(0,1)}\|\partial_t x_t^\star\|^2 + 2\frac{p}{\kappa}[2(p-1)+d]}\\
b &:= 2pr^{2p-1}\left[\sup_{t\in(0,1)}\|\partial_t x_t^\star\| + \frac{2(p-1)+d}{\epsilon r}\right],\\
\alpha_p &:= 2^{4p-2} \vee \left[1 + 2^{2p-1}\left(\frac{b}{\delta} + (1+2^{2p-1})\sup_{t\in[0,1]}\|x_t^\star\|^{2p}\right)\right].
\end{aligned}$$

Then the following hold:

$$\partial_t V_t^p(x) + \mathcal{L}_t V_t^p(x) \leqslant -\delta V_t^p(x) + b\,\mathbb{I}\{\|x - x_t^\star\| \leqslant r\}, \tag{43}$$

$$\mathbb{E}\left[\int_s^t V_u^p(X_{s,u}^x)\mathrm{d}u\right] = \int_s^t P_{s,u}V_u^p(x)\mathrm{d}u < +\infty, \tag{44}$$

$$P_{s,t}V_t^p(x) \leqslant e^{-\delta(t-s)}V_s^p(x) + \frac{b}{\delta}(1 - e^{-\delta(t-s)}), \tag{45}$$

$$\sup_{s\leqslant t}\mathbb{E}\left[1 + \|X_{s,t}^x\|^{2p}\right] \leqslant \alpha_p(1 + \|x\|^{2p}). \tag{46}$$

Proof. See section 4.1. □

Proposition 15 establishes regularity properties which are used in rigorously establishing the validity of the forward and backward equations in Proposition 16 and various manipulations in section 3.2. Although the topic of differentiability and other regularity properties of $x \mapsto P_{s,t}f(x)$ as in (47) is classical, we were not able to find in the literature results which give us exactly the conclusions we need under our assumptions, in particular allowing for time-inhomogeneity of $P_{s,t}f(x)$, and for $f(x)$ and $\nabla U_t(x)$ to be unbounded in x. The proof of Proposition 15 which we provide in section 4.3 to make the paper self-contained, does not exploit the elipticity of (42), which is why f is taken to be q-times differentiable on the left hand side of the implication in (47). This differentiability requirement propagates through our results, e.g., explaining why f is assumed twice differentiable in x in part 2) of Theorem 1. This restriction might be removed if existing results for elliptic diffusions, see for instance [6, Sec. 1.5, p.48], could be generalized to our setup, but that seems to involve a large amount of extra work which would further lengthen this paper.

Proposition 15. *For any given $p \geqslant 1$,*

$$f \in C_q^p(\mathbb{R}^d) \quad\Rightarrow\quad x \mapsto P_{s,t}f(x) \;\in\; C_q^p(\mathbb{R}^d), \quad \forall s \leqslant t,\ q = 1, 2, \tag{47}$$

$$f \in C_{1,2}^p([0,1]\times\mathbb{R}^d) \quad\Rightarrow\quad \begin{cases} (t,x) \mapsto |\partial_t f_t(x)| + |\mathcal{L}_t f_t(x)| \;\in\; C_{0,0}^{p+1/2}([0,1]\times\mathbb{R}^d), \\ (s,x) \mapsto \mathcal{L}_s P_{s,t} f_t(x) \;\in\; C_{0,0}^{p+1/2}([0,1]\times\mathbb{R}^d), \quad \forall t. \end{cases} \tag{48}$$

Proof. See section 4.2. □

Proposition 16. *For any $p \geqslant 1$, $f \in C_{1,2}^p([0,1]\times\mathbb{R}^d)$ and $\nu \in \mathcal{P}^{p+1/2}(\mathbb{R}^d)$, the following equalities hold:*

$$\partial_t \nu P_{s,t} f_t = \nu P_{s,t}\left(\partial_t f_t + \mathcal{L}_t f_t\right), \tag{49}$$

$$\partial_s P_{s,t} f_t(x) = -\mathcal{L}_s P_{s,t} f_t(x), \quad \forall x, \tag{50}$$

and for any fixed t, the map $(s,x) \mapsto P_{s,t}f_t(x)$ is a member of $C_{1,2}^{p+1/2}([0,1]\times\mathbb{R}^d)$.

Proof. See section 4.3. □

Before closing section 3.1, it is opportunte to discuss the derivation of the expectation formulae in (7)-(8) (see also Lemma 66 for the thermodynamic integration identity). Define

$$T_{s,t}f(x) := \mathbb{E}\left[f(X_{s,t}^x)\exp\left\{-\int_s^t \partial_u U_u(X_{s,u}^x)\mathrm{d}u\right\}\right].$$

To rigorously derive the path-integral representations of Z_1/Z_0 and $\pi_1(f)$ in (7)-(8) (note that we have already proved the first equality in (7) by Lemma 66), it is sufficient to verify the hypotheses on $T_{s,t}f$ of Lemma 17 below. Although we have not found an explicit verification of these hypotheses in the literature under exactly our assumptions (A1)-(5), we believe they are approachable using techniques similar to those in the proofs of Propositions 15 and 16. For example, a direct application of [17, Thm 2, p. 415] would require boundedness $|\partial_t U_t(\cdot)|$, but this condition seems not to be essential for the proof technique used there to work. A comprehensive account of the details would be very lengthy but not particularly interesting, and since we have already proved Lemma 66 and none of our main results actually rely on (51), we do not pursue this matter further.

Lemma 17. *Suppose that for any $p \geqslant 1$ and $f \in C_2^p(\mathbb{R}^d)$ there exists $q \geqslant 0$ such that for any t, $(s,x) \mapsto T_{s,t}f(x)$ is a member of $C_{1,2}^{p+q}([0,1] \times \mathbb{R}^d)$, and*

$$\partial_s T_{s,t}f(x) = -\mathcal{L}_s T_{s,t}f(x) + T_{s,t}f(x) \cdot \partial_s U_s(x), \quad \forall x.$$

Then

$$\frac{Z_1}{Z_0} = \pi_0 T_{0,1}1, \qquad \pi_1(f) = \frac{\pi_0 T_{0,1} f}{\pi_0 T_{0,1} 1}. \tag{51}$$

Proof. We shall prove

$$\frac{\partial}{\partial s} Z_s \pi_s T_{s,t} f = 0,$$

which implies

$$\pi_s T_{s,t} f = \frac{Z_t}{Z_s} \pi_t f, \quad \forall s \leqslant t,$$

and in turn (51).

We have

$$\begin{aligned}
\partial_s Z_s \pi_s T_{s,t} f &= \partial_s \int_{\mathbb{R}^d} \exp[-U_s(x)] T_{s,t} f(x) \mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \partial_s U_s(x) \exp[-U_s(x)] T_{s,t} f(x) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \exp[-U_s(x)][\mathcal{L}_s T_{s,t} f(x) - T_{s,t} f(x) \partial_s U_s(x)] \mathrm{d}x \\
&= 0,
\end{aligned}$$

where the interchange of differentiation and integration is justified by arguments similar to those in the proof of Lemma 66, using (A(1)), (A(2)), (A(4)), the assumption of the lemma and Lemma (14); and the final equality holds since $\pi_s \mathcal{L}_s T_{s,t} f = 0$. ☐

3.2 Poincaré inequalities, variance and bias bounds

3.2.1 The commutation relation

Lemma 18. *For any $p \geqslant 1$, $f \in C_2^p(\mathbb{R}^d)$, and $s \leqslant t$,*

$$\|\nabla P_{s,t} f\| \leqslant e^{-K(t-s)/\epsilon} P_{s,t} \|\nabla f\|. \tag{52}$$

Proof. By the mean value theorem,

$$f(X_{s,t}^x) - f(X_{s,t}^y) = \left\langle \nabla f(Z_{s,t}^{x,y}), X_{s,t}^x - X_{s,t}^y \right\rangle,$$

for some $Z_{s,t}^{x,y}$ on the line segment between $X_{s,t}^x$ and $X_{s,t}^y$. By Cauchy-Schwarz and Lemma 13,

$$|f(X_{s,t}^x) - f(X_{s,t}^y)| \leqslant \|\nabla f(Z_{s,t}^{x,y})\| \|X_{s,t}^x - X_{s,t}^y\| \leqslant \|\nabla f(Z_{s,t}^{x,y})\| e^{-K(t-s)/\epsilon} \|x - y\|,$$

hence

$$|P_{s,t} f(x) - P_{s,t} f(y)| \leqslant \mathbb{E}\left[|f(X_{s,t}^x) - f(X_{s,t}^y)|\right] \leqslant \mathbb{E}\left[\|\nabla f(Z_{s,t}^{x,y})\|\right] e^{-K(t-s)/\epsilon} \|x - y\|. \tag{53}$$

Now pick any $v \in \mathbb{R}^d$ such that $\|v\| = 1$ and set $y(n) := x + \frac{1}{n} v$. Our next step is to use dominated convergence to show:

$$\lim_{n \to \infty} \mathbb{E}\left[\|\nabla f(Z_t^{x,y(n)})\|\right] = \mathbb{E}\left[\|\nabla f(X_{s,t}^x)\|\right]. \tag{54}$$

Using Lemma 13, $Z_{s,t}^{x,y(n)} \to X_{s,t}^x$ a.s., hence $\|\nabla f(Z_{s,t}^{x,y(n)})\| \to \|\nabla f(X_{s,t}^x)\|$, a.s. By the assumption $f \in C_1^p(\mathbb{R}^d)$, there exists a constant $c < \infty$ such that

$$\|\nabla f(Z_{s,t}^{x,y})\| \leqslant c(1 + \|Z_{s,t}^{x,y}\|^{2p}),$$

and using the convexity of $a \mapsto a^{2p}$,

$$\begin{aligned}
\|\nabla f(Z_{s,t}^{x,y(n)})\| &\leqslant c\left[1+2^{2p-1}\left(\|Z_{s,t}^{x,y(n)}-X_{s,t}^{x}\|^{2p}+\|X_{s,t}^{x}\|^{2p}\right)\right]\\
&\leqslant c\left[1+2^{2p-1}\|X_{s,t}^{y(n)}-X_{s,t}^{x}\|^{2p}+2^{2p-1}\|X_{s,t}^{x}\|^{2p}\right]\\
&\leqslant c\left[1+2^{2p-1}\|x-y(n)\|^{2p}e^{-2pK(t-s)/\epsilon}+2^{2p-1}\|X_{s,t}^{x}\|^{2p}\right]\\
&\leqslant c\left[1+2^{2p-1}e^{-2pK(t-s)/\epsilon}+2^{2p-1}\|X_{s,t}^{x}\|^{2p}\right].
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\sup_n \|\nabla f(Z_{s,t}^{x,y(n)})\|\right] \leqslant c\left[1+2^{2p-1}e^{-2pK(t-s)/\epsilon}+2^{2p-1}\mathbb{E}\left[\|X_{s,t}^{x}\|^{2p}\right]\right] < +\infty,$$

using Lemma 14 for the final inequality. Thus we have proved that indeed (54) holds.

As $f \in C_1^p(\mathbb{R}^d)$, (47) implies $\nabla P_{s,t}f(x)$ exists and is continuous in x. Since $x - y(n) = v/n$, we have for some $z(n)$ between $y(n)$ and x,

$$P_{s,t}f(y(n)) - P_{s,t}f(x) = \frac{1}{n}\left\langle \nabla P_{s,t}f(z(n)), v\right\rangle,$$

so by the continuity of $\nabla P_{s,t}f$ we then obtain from (53) and (54):

$$|\langle \nabla P_{s,t}f(x), v\rangle| = \lim_n \frac{|P_{s,t}f(x) - P_{s,t}f(y(n))|}{\|x-y(n)\|} \leqslant e^{-K(t-s)/\epsilon}P_{s,t}(\|\nabla f\|)(x).$$

Taking $v = \nabla P_{s,t}f(x)/\|\nabla P_{s,t}f(x)\|$ completes the proof. ☐

Remark 19. It can be shown that in fact the strong log-concavity assumption (A4) is necessary for the statement of Lemma 18 to hold. Indeed, when that statement does hold, the same line of argument as [25, Lem. 1.2 or 1.3] shows that the Bakry-Émery criterion holds for U_t with constant K, uniformly in t, i.e. for all $f \in C_2^p(\mathbb{R}^d)$,

$$\inf_{t\in[0,1]} \left\langle \nabla^{(2)}U_t \cdot \nabla f, \nabla f\right\rangle + \|\nabla^{(2)} f\|_{\text{H.S.}}^2 \geqslant K\|\nabla f\|^2.$$

So for an arbitrary $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, choosing $f(x) = \sum_{i=1}^d v_i x_i$ gives $\nabla f = v$ and $\|\nabla^{(2)} f\|_{\text{H.S.}}^2 = 0$, hence

$$\inf_{t\in[0,1]} \left\langle \nabla^{(2)}U_t \cdot v, v\right\rangle \geqslant K\|v\|^2,$$

which is exactly (A4).

3.2.2 Poincaré inequalties

Lemma 20. *For any $s \leqslant t$ and $f \in C_2^p(\mathbb{R}^d)$,*

$$P_{s,t}(f^2) - (P_{s,t}f)^2 \leqslant \frac{1}{K}(1 - e^{-2K(t-s)/\epsilon})P_{s,t}(\|\nabla f\|^2). \tag{55}$$

Proof. Consider t fixed and write $g(u,x) = (P_{u,t}f(x))^2$. By Proposition 16, $(u,x) \mapsto P_{u,t}f_t(x)$ is a member of $C_{1,2}^{p+1/2}([0,1]\times\mathbb{R}^d)$, so $g \in C_{1,2}^{2p+1}([0,1]\times\mathbb{R}^d)$. We then may apply (49) with $\nu = \delta_x$ to obtain:

$$\begin{aligned}
\partial_u P_{s,u}\left[(P_{u,t}f)^2\right] &= \partial_u P_{s,u} g_u\\
&= P_{s,u}\left[\frac{\partial g}{\partial u} + \mathcal{L}_u g_u\right]\\
&= -2P_{s,u}\left[(P_{u,t}f)(\mathcal{L}_u P_{u,t}f)\right] + P_{s,u}\left[\mathcal{L}_u (P_{u,t}f)^2\right]\\
&= 2\epsilon^{-1}P_{s,u}(\|\nabla P_{u,t}f\|^2)\\
&\leqslant 2\epsilon^{-1}e^{-2K(t-u)/\epsilon}P_{s,t}(\|\nabla f\|^2),
\end{aligned}$$

where the penultimate equality is an application of (50), the final equality holds due to the well known Carré du champ identity: $\mathcal{L}_u(P_{u,t}f)^2 - 2(P_{u,t}f)(\mathcal{L}_u P_{u,t}f) = 2\epsilon^{-1}\|\nabla P_{u,t}f\|^2$, and the inequality is due to Lemma 18 and Jensen's inequality. Integrating w.r.t. to u from s to t gives (55). ☐

Remark 21. It is well known that under (A4), each π_t satisfies a Poincaré inequality with constant K, that is

$$\mathrm{var}_{\pi_t}[f] \leqslant \frac{1}{K}\pi_t(\|\nabla f\|^2), \tag{56}$$

for f in some class of suitably smooth functions. We have particular interest in the case $f \in C_2^p(\mathbb{R}^d)$, and one can verify that indeed (56) holds for that class of functions using Lemma 20; for example considering π_0, assume that $U_t = U_0$ for all $t \in (0,1]$, so that $P_{s,t}$ becomes time-homogeneous and $\pi_0 P_{0,t} = \pi_0$. Then with $s = 0$, $t = 1$, integrating (55) w.r.t. π_0 gives

$$\mathrm{var}_{\pi_0}[f] \leqslant \mathrm{var}_{\pi_0}[P_{0,1}f] + \frac{1}{K}(1 - e^{-2K/\epsilon})\pi_0(\|\nabla f\|^2),$$

and $\mathrm{var}_{\pi_0}[P_{0,1}f] \to 0$ as $\epsilon \to 0$ by standard results for the time-homogeneous Langevin diffusion (a particular rate of convergence for $\mathrm{var}_{\pi_0}[P_{0,1}f] \to 0$ is not need for this computation).

Lemma 22. *Fix $p \geqslant 1$. If for some given $\nu \in \mathcal{P}^{2p}(\mathbb{R}^d)$ and constant $K_\nu > 0$,*

$$\mathrm{var}_\nu[f] \leqslant \frac{1}{K_\nu}\nu(\|\nabla f\|^2), \quad \forall f \in C_2^p(\mathbb{R}^d), \tag{57}$$

then for all $s \leqslant t$,

$$\mathrm{var}_{\nu P_{s,t}}[f] \leqslant \left[(1 - e^{-2K(t-s)/\epsilon})\frac{1}{K} + e^{-2K(t-s)/\epsilon}\frac{1}{K_\nu}\right]\nu P_{s,t}(\|\nabla f\|^2), \quad \forall f \in C_2^p(\mathbb{R}^d).$$

Proof. Since $\nu \in \mathcal{P}^{2p}(\mathbb{R}^d)$ we are guaranteed $\nu(\|\nabla f\|^2) < +\infty$, and using Lemma 14, $\nu P_{s,t}(\|\nabla f\|^2) < +\infty$. Integrating (55) w.r.t. ν gives

$$\nu P_{s,t}(f^2) - \nu[(P_{s,t}f)^2] \leqslant \frac{1}{K}(1 - e^{-2K(t-s)/\epsilon})\nu P_{s,t}(\|\nabla f\|^2).$$

By Proposition 15, if $f \in C_2^p(\mathbb{R}^d)$ then $P_{s,t}f \in C_2^p(\mathbb{R}^d)$, so under the hypotheses of the lemma, the inequality (57) holds with f replaced by $P_{s,t}f$. This observation, together with Lemma 18 and Jensen's inequality give:

$$\begin{aligned}
\mathrm{var}_{\nu P_{s,t}}[f] &\leqslant \mathrm{var}_\nu[P_{s,t}f] + \frac{1}{K}(1 - e^{-2K(t-s)/\epsilon})\nu P_{s,t}(\|\nabla f\|^2) \\
&\leqslant \frac{1}{K_\nu}\nu(\|\nabla P_{s,t}f\|^2) + \frac{1}{K}(1 - e^{-2K(t-s)/\epsilon})\nu P_{s,t}(\|\nabla f\|^2) \\
&\leqslant \frac{1}{K_\nu}\nu P_{s,t}(\|\nabla f\|^2)e^{-2K(t-s)/\epsilon} + \frac{1}{K}(1 - e^{-2K(t-s)/\epsilon})\nu P_{s,t}(\|\nabla f\|^2).
\end{aligned}$$

☐

3.2.3 Variance bounds

Lemma 23. *Fix $p \geqslant 1$ and $s \leqslant t$. If for some given $\nu \in \mathcal{P}^{2p}(\mathbb{R}^d)$ and a strictly positive, continuous function $\kappa_\nu : u \in [s,t] \mapsto \kappa_\nu(u) \in \mathbb{R}^+$,*

$$\mathrm{var}_{\nu P_{s,u}}[f] \leqslant \frac{1}{\kappa_\nu(u)}\nu P_{s,u}(\|\nabla f\|^2), \quad \forall f \in C_2^p(\mathbb{R}^d),\ u \in [s,t],$$

then

$$\mathrm{var}_{\nu P_{s,u}}[P_{u,t}f] \leqslant \exp\left[-\frac{2}{\epsilon}\int_u^t \kappa_\nu(\tau)\mathrm{d}\tau\right]\mathrm{var}_{\nu P_{s,t}}[f], \quad \forall f \in C_2^p(\mathbb{R}^d),\ u \in [s,t].$$

Proof. Arguing similarly to the proof of Lemma 20, the map $(u, x) \mapsto (P_{u,t}f(x))^2$ is a member of $C^{2p+1}_{1,2}([0,1]\times \mathbb{R}^d)$ and $P_{u,t}f \in C^p_2(\mathbb{R}^d)$. Applying (49) and (50),

$$\begin{aligned}
\partial_u \mathrm{var}_{\nu P_{s,u}}[P_{u,t}f] &= \partial_u \nu P_{s,u}[(P_{u,t}f)^2] \\
&= \nu P_{s,u}\mathcal{L}_u[(P_{u,t}f)^2] - 2\nu P_{s,u}[(P_{u,t}f)(\mathcal{L}_u P_{u,t}f)] \\
&= \frac{2}{\epsilon}\nu P_{s,u}(\|\nabla P_{u,t}f\|^2) \\
&\geqslant \frac{2}{\epsilon}\kappa_\nu(u)\mathrm{var}_{\nu P_{s,u}}[P_{u,t}f],
\end{aligned}$$

where the inequality holds by the hypothesis of the lemma. With $\beta(u) := \mathrm{var}_{\nu P_{s,u}}[P_{u,t}f]$ we have shown

$$\beta'(u) \geqslant \frac{2}{\epsilon}\kappa_\nu(u)\beta(u),$$

so

$$u \mapsto \beta(u)\exp\left[-\frac{2}{\epsilon}\int_s^u \kappa_\nu(\tau)\mathrm{d}\tau\right]$$

is a non-decreasing function on $[s, t]$, which implies

$$\beta(u) \leqslant \beta(t)\exp\left[-\frac{2}{\epsilon}\int_u^t \kappa_\nu(\tau)\mathrm{d}\tau\right],$$

as required. □

3.2.4 Bias bounds

Introduce

$$W^{(p)}(\nu,\bar{\nu}) := \inf_{\gamma\in\Gamma(\nu,\bar{\nu})}\int_{\mathbb{R}^{2d}} \left(1 + \|x\|^{2p} \vee \|y\|^{2p}\right)\|x-y\|\gamma(\mathrm{d}x,\mathrm{d}y),$$

where $\Gamma(\nu,\bar{\nu})$ is the set of all couplings of two probability measures $\nu, \bar{\nu}$ on $\mathcal{B}(\mathbb{R}^d)$.

Lemma 24. *For any $p \geqslant 1$, $f \in C^p_2(\mathbb{R}^d)$ and $\nu, \bar{\nu} \in \mathcal{P}^p(\mathbb{R}^d)$,*

$$|\nu P_{s,t}f - \bar{\nu}P_{s,t}f| \leqslant \alpha_p\|\nabla f\|_p e^{-K(t-s)/\epsilon}W^{(p)}(\nu,\bar{\nu})$$

where α_p is the constant from Lemma 14, which depends on ϵ, K, p, d, $\sup_t \|\partial_t x^\star_t\|$ and $\sup_t \|x^\star_t\|$.

Proof. Pick any $x, y \in \mathbb{R}^d$ and $s \leqslant t$. Then by the mean value theorem there exists a point z on the line segment between x and y such that,

$$\begin{aligned}
|P_{s,t}f(x) - P_{s,t}f(y)| &= |\langle \nabla P_{s,t}f(z), x-y\rangle| \\
&\leqslant \|\nabla P_{s,t}f(z)\|\|x-y\| \\
&\leqslant e^{-K(t-s)/\epsilon}P_{s,t}(\|\nabla f\|)(z)\|x-y\| \\
&\leqslant \|\nabla f\|_p e^{-K(t-s)/\epsilon}(1 + \mathbb{E}[\|X^z_{s,t}\|^{2p}])\|x-y\| \\
&\leqslant \alpha_p\|\nabla f\|_p e^{-K(t-s)/\epsilon}\left[1 + \|x\|^{2p} \vee \|y\|^{2p}\right]\|x-y\|,
\end{aligned}$$

where the second inequality is due to Lemma 18, and the fourth inequality uses Lemma 14 and the fact $\|z\| \leqslant \|x\| \vee \|y\|$. The proof is completed by noting:

$$|\nu P_{s,t}f - \bar{\nu}P_{s,t}f| \leqslant \int |P_{s,t}f(x) - P_{s,t}f(y)|\gamma(\mathrm{d}x,\mathrm{d}y), \quad \forall\gamma \in \Gamma(\nu,\bar{\nu}).$$

□

Lemma 25. *For any $p \geqslant 1$,*

$$\sup_t \int_{\mathbb{R}^d} \|x\|^{2p} \pi_t(\mathrm{d}x) < +\infty, \tag{58}$$

and for any $f \in C_2^p(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \sup_s \left| \partial_s \left\{ \frac{\exp[-U_s(x)]}{Z_s} P_{s,t} f(x) \right\} \right| \mathrm{d}x < +\infty. \tag{59}$$

Proof. We have

$$\begin{aligned}
&\left| \partial_s \left\{ \frac{\exp[-U_s(x)]}{Z_s} P_{s,t} f(x) \right\} \right| \\
&= \frac{\exp[-U_s(x)]}{Z_s} \left| \phi_s(x) P_{s,t} f(x) - \mathcal{L}_s P_{s,t} f(x) \right| \\
&\leqslant \frac{\exp[-U_s(x)]}{Z_s} \left[|\phi_s(x)| |P_{s,t} f(x)| + |\mathcal{L}_s P_{s,t} f(x)| \right]
\end{aligned}$$

Under (A2) and (A4), for all $s \in [0,1]$ and $x \in \mathbb{R}^d$,

$$\inf_t U_t(x_t^\star) + \left(\|x\| - \inf_t \|x_t^\star\| \right)^2 \frac{K}{2} \leqslant U_s(x) \leqslant \frac{L}{2} \left(\|x\| + \sup_t \|x_t^\star\| \right)^2 + \sup_t U_t(x_t^\star), \tag{60}$$

where the infima and suprema are finite, since by Lemma 68, $t \mapsto \|x_t^\star\|$ is continuous on $[0,1]$, and $U_t(x)$ is continous in (t,x) by (A1). It follows from (60) that $\inf_t Z_t > 0$ and $\sup_s \exp[-U_s(x)] \leqslant \exp[-c_1\|x\|^2 + c_2]$ for some finite constants $c_1, c_2 > 0$, which implies (58). Also, since $U \in C_{1,2}^{p_0}([0,1] \times \mathbb{R}^d)$ under (A1), it follows from (60) and Lemma 66 that $(t,x) \mapsto \phi_t(x)$ is a member of $C_{0,2}^{p_0}([0,1] \times \mathbb{R}^d)$. Since $f \in C_2^p(\mathbb{R}^d)$, it follows from Proposition 16 that $(s,x) \mapsto P_{s,t} f(x)$ is a member of $C_{1,2}^{p+1/2}([0,1] \times \mathbb{R}^d)$ and from Proposition 15 that $(s,x) \mapsto \mathcal{L}_s P_{s,t} f(x)$ is a member of $C_{0,0}^{p+1/2}([0,1] \times \mathbb{R}^d)$. These observations together imply (59). ☐

Lemma 26. *For any $p \geqslant 1$ and $f \in C_2^p(\mathbb{R}^d)$,*

$$|\pi_0 P_{0,t} f - \pi_t f| \leqslant \sup_{s \in [0,t]} \mathrm{var}_{\pi_s}[\phi_s]^{1/2} \mathrm{var}_{\pi_t}[f]^{1/2} \frac{\epsilon}{K} (1 - e^{-Kt/\epsilon}).$$

Proof. Write

$$\pi_t f - \pi_0 P_{0,t} f = \int_0^t \partial_s \pi_s P_{s,t} f \mathrm{d}s, \tag{61}$$

and

$$\begin{aligned}
\partial_s \pi_s P_{s,t} f &= \int_{\mathbb{R}^d} \partial_s \left[\frac{\exp[-U_s(x)]}{Z_s} P_{s,t} f(x) \right] \mathrm{d}x \\
&= -\pi_s[\phi_s P_{s,t} f] - \pi_s \mathcal{L}_s P_{s,t} f \\
&= -\pi_s[(\phi_s - \pi_s \phi_s)(P_{s,t} f - \pi_s P_{s,t} f)],
\end{aligned}$$

where the first equality is validated by Lemma 25; the second equality holds by the definition of ϕ_s, see (11), and Proposition 16; and the third equality holds because by Lemma 66 $\pi_s \phi_s = 0$, and $\mathcal{L}_s$ is the generator of a Langevin diffusion with invariant distribution π_s. Therefore

$$\begin{aligned}
|\partial_s \pi_s P_{s,t} f|^2 &\leqslant \mathrm{var}_{\pi_s}[\phi_s] \mathrm{var}_{\pi_s}[P_{s,t} f] \\
&\leqslant \mathrm{var}_{\pi_s}[\phi_s] \mathrm{var}_{\pi_t}[f] e^{-2K(t-s)/\epsilon}
\end{aligned}$$

where Cauchy-Schwartz and Lemmas 22 and 23 with $\nu = \pi_s$ have been applied, noting Remark 21. Plugging this bound into (61) and integrating completes the proof. ☐

4 Proofs and supporting results for section 3

4.1 Proof of Lemma 14

Proof of Lemma 14. We have

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\|x-x_t^\star\|^{2p} &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d}(x_j-x_{t,j}^\star)^2\right)^p = 2p\|x-x_t^\star\|^{2(p-1)}(x_i-x_{t,i}^\star)\\
\frac{\partial^2}{\partial x_i^2}\|x-x_t^\star\|^{2p} &= 4p(p-1)\|x-x_t^\star\|^{2(p-2)}(x_i-x_{t,i}^\star)^2+2p\|x-x_t^\star\|^{2(p-1)}\\
\partial_t\|x-x_t^\star\|^{2p} &= p\|x-x_t^\star\|^{2(p-1)}2\sum_{j=1}^{d}(x_j-x_{t,j}^\star)\left(-\partial_t x_{t,j}^\star\right)\\
&= -2p\|x-x_t^\star\|^{2(p-1)}\langle x-x_t^\star,\partial_t x_t^\star\rangle
\end{aligned}
$$

and via Lemma 67, (A4) implies

$$\langle\nabla U_t(x)\,,\,x-x_t^\star\rangle\geqslant\frac{K}{2}\|x-x_t^\star\|^2.$$

Therefore

$$
\begin{aligned}
-\langle\nabla U_t(x)\,,\,\nabla V_t^p(x)\rangle &= -2p\|x-x_t^\star\|^{2(p-1)}\langle\nabla U_t(x)\,,\,x-x_t^\star\rangle\\
&\leqslant -Kp\|x-x_t^\star\|^{2p},\\
\Delta V_t^p(x) &= 4p(p-1)\|x-x_t^\star\|^{2(p-2)}\sum_{i=1}^{d}(x_i-x_{t,i}^\star)^2+2dp\|x-x_t^\star\|^{2(p-1)}\\
&= 2p\left(2(p-1)+d\right)\|x-x_t^\star\|^{2(p-1)},\\
|\partial_t V_t^p(x)| &\leqslant 2p\|x-x_t^\star\|^{2p-1}\|\partial_t x_t^\star\|\\
&\leqslant 2p\|x-x_t^\star\|^{2p-1}c,
\end{aligned}
$$

where in the final inequality, $c:=\sup_{t\in(0,1)}\|\partial_t x_t^\star\|$ is finite by Lemma 68. Combining the above we have

$$
\begin{aligned}
&\epsilon\partial_t V_t^p(x)(x)+\epsilon\mathcal{L}_t V_t^p(x)\\
&\leqslant -Kp\|x-x_t^\star\|^{2p}+2p\|x-x_t^\star\|^{2p-1}\left[\epsilon c+\left(2(p-1)+d\right)\|x-x_t^\star\|^{-1}\right]\\
&= -(Kp-\kappa)\|x-x_t^\star\|^{2p}-\kappa\|x-x_t^\star\|^{2p}+2p\|x-x_t^\star\|^{2p-1}\left[\epsilon c+\left(2(p-1)+d\right)\|x-x_t^\star\|^{-1}\right]\\
&= -(Kp-\kappa)\|x-x_t^\star\|^{2p}-\|x-x_t^\star\|^{2p}\left(\kappa-2p\left[\frac{\epsilon c}{\|x-x_t^\star\|}+\frac{2(p-1)+d}{\|x-x_t^\star\|^2}\right]\right).
\end{aligned}
$$

Hence

$$\partial_t V_t^p(x)+\mathcal{L}_t V_t^p(x)\leqslant-\delta\|x-x_t^\star\|^{2p}+b\mathbb{I}\{\|x-x_t^\star\|\leqslant r\},$$

where

$$
\begin{aligned}
\delta &:= \epsilon^{-1}(Kp-\kappa),\\
r &:= \sup\left\{a>0\,:\,\frac{\epsilon c}{a}+\frac{2(p-1)+d}{a^2}\geqslant\frac{\kappa}{2p}\right\},\\
b &:= 2pr^{2p-1}\left[c+\frac{2(p-1)+d}{\epsilon r}\right].
\end{aligned}
$$

Solving the quadratic inequality in the expression for r completes the proof of (43).

In the remainder of the proof of the lemma, we write

$$
\begin{aligned}
V^p(t,x) &\equiv V_t^p(x)=\|x-x_t^\star\|^{2p},\\
\mathcal{L}V^p(t,x) &\equiv \partial_t V_t^p(x)+\mathcal{L}_t V_t^p(x).
\end{aligned}
$$

Fix $s \in [0,1]$ and $x \in \mathbb{R}^d$. Define $T_m := \inf\{t \geqslant s : \|X^x_{s,t}\| > m\}$, the dependence of T_m on x and s is not shown in the notation. By non-explosivity of the process, $T_m \to \infty$, a.s.

By Dynkin's formula [23, Lem. 3.2, p.72] and (43), for any m such that $\|x\| \leqslant m$,

$$\begin{aligned}
&\mathbb{E}[V^p(T_m \wedge t, X^x_{s,T_m\wedge t})] + \delta\mathbb{E}\left[\int_s^{T_m\wedge t} V^p(u, X^x_{s,u})\mathrm{d}u\right] \\
&= V^p(s,x) + \mathbb{E}\left[\int_s^{T_m\wedge t} \mathcal{L}V^p(u, X^x_{s,u})\mathrm{d}u\right] + \delta\mathbb{E}\left[\int_s^{T_m\wedge t} V^p(u, X^x_{s,u})\mathrm{d}u\right] \\
&\leqslant V^p(s,x) + b(t-s) < +\infty,
\end{aligned}$$

hence $\mathbb{E}\left[\int_s^t V^p(u, X^x_{s,u})du\right] = \lim_m \mathbb{E}\left[\int_s^{T_m\wedge t} V^p(u, X^x_{s,u})du\right] < +\infty$, where the limit exists by monotone convergence. Also, by Tonelli's theorem $\mathbb{E}\left[\int_s^t V^p(u, X^x_{s,u})\mathrm{d}u\right] = \int_s^t P_{s,u}V^p_u(x)\mathrm{d}u$. This completes the proof of (44).

Applying Fatou, (43) and (44) we have

$$\begin{aligned}
\mathbb{E}[V^p(t, X^x_{s,t})] &= \mathbb{E}[\liminf_m V^p(T_m \wedge t, X^x_{s,T_m\wedge t})] \leqslant \liminf_m \mathbb{E}[V^p(T_m \wedge t, X^x_{s,T_m\wedge t})] \\
&\leqslant \liminf_m \left\{ V^p(s,x) - \delta\mathbb{E}\left[\int_s^{T_m\wedge t} V^p(u, X^x_{s,u})\mathrm{d}u\right] + \mathbb{E}\left[\int_s^{T_m\wedge t} b\mathbb{I}[\|X^x_{s,u}\| \leqslant r]\mathrm{d}u\right]\right\} \\
&= V^p(s,x) - \delta\mathbb{E}\left[\int_s^t V^p(u, X^x_{s,u})\mathrm{d}u\right] + \mathbb{E}\left[\int_s^t b\mathbb{I}[\|X^x_{s,u}\| \leqslant r]\mathrm{d}u\right],
\end{aligned}$$

hence

$$P_{s,t}V^p_t(x) \leqslant V^p_s(x) - \delta\int_s^t P_{s,u}V^p_u(x)\mathrm{d}u + b(t-s).$$

This inequality is solved to give (45).

To establish (46), we have by (45),

$$\begin{aligned}
1 + \mathbb{E}\left[\|X^x_{s,t}\|^{2p}\right] &\leqslant 1 + 2^{2p-1}\mathbb{E}\left[V^p_t(X^x_{s,t})\right] + 2^{2p-1}\|x^\star_t\|^{2p} \\
&\leqslant 1 + 2^{2p-1}V^p_s(x) + 2^{2p-1}\frac{b}{\delta} + 2^{2p-1}\|x^\star_t\|^{2p} \\
&\leqslant 2^{4p-2}\|x\|^{2p} + 1 + 2^{2p-1}\frac{b}{\delta} + 2^{2p-1}(1+2^{2p-1})\sup_{u\in[0,1]}\|x^\star_u\|^{2p} \\
&\leqslant \alpha_p(1+\|x\|^{2p}),
\end{aligned}$$

where $\sup_{u\in[0,1]}\|x^\star_u\|^{2p}$ is finite since by Lemma 68 $t \mapsto x^\star_t$ is continuous on $[0,1]$, and α_p is as in the statement of the Lemma. The proof is complete. □

4.2 Proof and supporting results for Proposition 15

Lemma 27. *For any $p \geqslant 1$, and $\nu \in \mathcal{P}^p(\mathbb{R}^d)$, the following condition holds:*

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\sup_{t\in[s,1]} \|X^x_{s,t}\|^{2p}\right]\nu(\mathrm{d}x) < +\infty, \tag{62}$$

and for any $f \in C^p_{0,0}([0,1]\times\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\mathbb{E}[f(t, X^x_{s,t})]\nu(\mathrm{d}x)$ is continuous in s and t.

Proof. By assumption $\sup_t |f(t,x)| \leqslant c(1+\|x\|^{2p})$, so the assumption $\nu \in \mathcal{P}^p(\mathbb{R}^d)$ combined with equation (44) of Lemma 14 guarantees that $\mathbb{E}[f(t, X^x_{s,t})]$ is integrable w.r.t. ν. As noted in section 3.1, $X^x_{s,t}$ is continuous in t, a.s., and f is continuous by assumption, so to establish the continuity in t of $\int_{\mathbb{R}^d}\mathbb{E}[f(t, X^x_{s,t})]\nu(\mathrm{d}x)$ by an application of dominated convergence, it suffices to show (62). From (42),

$$\sup_{t\in[s,1]}\|X^x_{s,t}\| \leqslant \|x\| + \epsilon^{-1}\int_s^1 \|\nabla U_u(X^x_{s,u})\|\mathrm{d}u + \sqrt{2\epsilon^{-1}}\sup_{t\in[s,1]}\|B_t\|.$$

Using (A3), the fact that $s \in [0,1]$, Jensen's inequality, the convexity of $a \mapsto a^{2p}$, and equation (44) of Lemma 14,

$$\begin{aligned} \mathbb{E}\left[\left(\int_s^1 \|\nabla U_u(X_{s,u}^x)\|\mathrm{d}u\right)^{2p}\right] &\leqslant L^{2p}2^{2p-1}\mathbb{E}\left[\int_s^t 1 + \|X_{s,u}^x\|^{2p}\mathrm{d}u\right], \\ &\leqslant L^{2p}2^{2p-1}\alpha_p(1+\|x\|^{2p}). \end{aligned} \tag{63}$$

The integral of (63) with respect to ν is finite due to the assumption $\nu \in \mathcal{P}^p(\mathbb{R}^d)$. The expected value of $\sup_{t\in[s,1]}\left\|\int_s^t \mathrm{d}B_u\right\|^{2p}$ is finite by standard results for Brownian motion, e.g. [22, Prob. 3.29 and Rem. 3.30, Ch. 3, p. 166], and does not depend on x. Therefore (62) holds as required so $\mathbb{E}[f(t, X_{s,t}^x)]$ is continous in t. The proof of continuity in s is very similar so the details are omitted. ☐

The following notations are in force throughout the remainder of section 4.2. For a matrix A and vector b of appropriate sizes we write $A \circ b$ for the usual matrix vector product. We introduce the shorthands:

$$F_{s,t}^x[i] := -\frac{1}{\epsilon}\frac{\partial U_t}{\partial x_i}(X_{s,t}^x), \qquad DF_{s,t}^x[i,j] := -\frac{1}{\epsilon}\frac{\partial^2 U_t}{\partial x_i \partial x_j}(X_{s,t}^x), \quad D^2F_{s,t}^x[i,j,k] := -\frac{1}{\epsilon}\frac{\partial^3 U_t}{\partial x_i \partial x_j \partial x_k}(X_{s,t}^x).$$

Thus $F_{s,t}^x$ is a random vector of length d, and $DF_{s,t}^x$ is a random $d \times d$ matrix.

Proposition 28. *Write (42) component-wise as*

$$X_{s,t}^x[i] = x[i] + \int_s^t F_{s,u}^x[i]\mathrm{d}u + \sqrt{2\epsilon^{-1}}\int_s^t \mathrm{d}B_u[i], \quad t \in [s,1],\ i \in \{1,\dots,d\}. \tag{64}$$

Then for $(i,j,k) \in \{1,\dots,d\}^3$ and $t \in [s,1]$, the solutions of:

$$\zeta_{s,t}^x[i,j] = \mathbb{I}[i=j] + \int_s^t \left\langle DF_{s,u}^x[\cdot,i]\,,\, \zeta_{s,u}^x[\cdot,j]\right\rangle \mathrm{d}u, \tag{65}$$

$$\eta_{s,t}^x[i,j,k] = \int_s^t \left\langle D^2F_{s,u}^x[\cdot,\cdot,i] \circ \zeta_{s,u}^x[\cdot,k]\,,\, \zeta_{s,u}^x[\cdot,j]\right\rangle + \left\langle DF_{s,u}^x[\cdot,i]\,,\, \eta_{s,u}^x[\cdot,j,k]\right\rangle \mathrm{d}u, \tag{66}$$

satisfy

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\zeta_{s,t}^x[i,j] - n\{X_{s,t}^x[i] - X_{s,t}^{y(n)}[i]\}\right)^2\right] = 0, \quad \text{with} \quad y(n) := x + n^{-1}e_j \tag{67}$$

and

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\eta_{s,t}^x[i,j,k] - n\{\zeta_{s,t}^x[i,j] - \zeta_{s,t}^{y(n)}[i,j]\}\right)^2\right] = 0, \quad \text{with} \quad y(n) := x + n^{-1}e_k. \tag{68}$$

Moreover $\zeta_{s,t}^x[i,j]$ and $\eta_{s,t}^x[i,j,k]$ are mean-square continuous in x.

Proof. Under (A2), (A3) and (A6), the existence of random functions $\zeta_{s,t}^x[i,j]$ and $\eta_{s,t}^x[i,j,k]$ which satisfy (67)-(68) and are mean-square continuous in x is a direct application of [17, Thm. 2, p. 410]. The fact that $\zeta_{s,t}^x[i,j]$ and $\eta_{s,t}^x[i,j,k]$ satisfy (65)-(66), i.e. the equations obtained by formally differentiating in (71), is a classical fact noted for example by [23, Thm. 5.10, p.166], see also [24, Thm. 3.1, p. 218]. ☐

Lemma 29.
1) there exists a finite constant c_1 such that $\sup_x \sup_{0\leqslant s\leqslant t\leqslant 1}\|\zeta_{s,t}^x\|_{\text{H.S.}} \leqslant c_1$, a.s.,
2) for any $s \leqslant t$ and $f \in C_1^p(\mathbb{R}^d)$, $P_{s,t}f(x)$ is differentiable in x, the following identity holds:

$$\frac{\partial P_{s,t}f}{\partial x_i}(x) = \mathbb{E}\left[\left\langle \nabla f(X_{s,t}^x)\,,\, \zeta_{s,t}^x[\cdot,i]\right\rangle\right], \tag{69}$$

and $\nabla P_{s,t}f(x)$ is continuous in x, s and t.

Lemma 30.

1) there exists a finite constant c_2 such that $\sup_x \sup_{0\leqslant s\leqslant t\leqslant 1} \|\eta^x_{s,t}\|_{\text{H.S.}} \leqslant c_2$, a.s.

2) for any $s \leqslant t$ and $f \in C^p_2(\mathbb{R}^d)$, $P_{s,t}f(x)$ is twice differentiable in x, the following identity holds:

$$\frac{\partial^2 P_{s,t}f}{\partial x_i \partial x_j}(x) = \mathbb{E}\left[\left\langle \nabla^{(2)} f(X^x_{s,t}) \circ \zeta^x_{s,t}[\cdot,j]\,,\, \zeta^x_{s,t}[\cdot,i]\right\rangle\right] + \mathbb{E}\left[\left\langle \nabla f(X^x_{s,t})\,,\, \eta^x_{s,t}[\cdot,i,j]\right\rangle\right], \tag{70}$$

and $\nabla^{(2)}P_{s,t}f(x)$ is continuous in x, s and t.

Proof of Lemma 29. Throughout the proof, c is a finite constant whose value may change on each appearance.

For part 1), it follows from (65) that

$$\begin{aligned}
\|\zeta^x_{s,t}[\cdot,j]\|^2 &\leqslant 2 + 2\sum_{i=1}^d \left(\int_s^t \left\langle DF^x_{s,u}[\cdot,i]\,,\, \zeta^x_{s,u}[\cdot,j]\right\rangle \mathrm{d}u\right)^2 \\
&\leqslant 2 + 2\sum_{i=1}^d \left(\int_s^t \|DF^x_{s,u}[\cdot,i]\|\|\zeta^x_{s,u}[\cdot,j]\|\mathrm{d}u\right)^2 \\
&\leqslant 2 + 2(t-s)\sum_{i=1}^d \int_s^t \|DF^x_{s,u}[\cdot,i]\|^2\|\zeta^x_{s,u}[\cdot,j]\|^2\mathrm{d}u \\
&= 2 + 2(t-s)\int_s^t \|DF^x_{s,u}\|^2_{\text{H.S.}}\|\zeta^x_{s,u}[\cdot,j]\|^2\mathrm{d}u \\
&\leqslant 2 + c(t-s)\int_s^t \|\zeta^x_{s,u}[\cdot,j]\|^2\mathrm{d}u.
\end{aligned}$$

where the first inequality uses the fact that for any $a, b \in \mathbb{R}^d$ $\|a+b\|^2 \leqslant 2(\|a\|^2 + \|b\|^2)$; the second inequality uses Cauchy-Schwartz; the third inequality uses Jensen's inequality; the final inequality uses (A2), and there c is a finite constant depending on L and ϵ but independent of j, x. It then follows from Gronwall's lemma that

$$\|\zeta^x_{s,t}[\cdot,j]\|^2 \leqslant 2\exp[c(t-s)^2],$$

the r.h.s. of which is a finite constant independent of x and j. The claim of part 1) then holds.

Considering now that s, t are fixed, we de-clutter the notation by writing

$$X^x \equiv X^x_{s,t}, \qquad \zeta^x \equiv \zeta^x_{s,t}.$$

Fix any $f \in C^p_2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and set $y(n) := x + n^{-1}e_i$. To establish the identity in part 2) we shall show that

$$\lim_{n\to\infty} \frac{P_{s,t}f(x) - P_{s,t}f(y(n))}{n^{-1}} = \mathbb{E}\left[\langle \nabla f(X^x)\,,\, \zeta^x[\cdot,i]\rangle\right].$$

By the mean value theorem, let us introduce a random variable $Z^{x,y(n)}$, valued on the line segment bewteen X^x and $X^{y(n)}$ such that:

$$f(X^x) - f(X^{y(n)}) = \left\langle \nabla f(Z^{x,y(n)})\,,\, X^x - X^{y(n)}\right\rangle, \quad a.s.$$

Then using Cauchy-Schwartz we have

$$\begin{aligned}
&\left|\frac{P_{s,t}(x) - P_{s,t}(y(n))}{1/n} - \mathbb{E}\left[\langle \nabla f(X^x)\,,\, \zeta^x[\cdot,i]\rangle\right]\right| \\
=\ &\left|\mathbb{E}\left[\frac{f(X^x) - f(X^{y(n)})}{1/n} - \langle \nabla f(X^x)\,,\, \zeta^x[\cdot,i]\rangle\right]\right| \\
=\ &\left|\mathbb{E}\left[\left\langle \nabla f(Z^{x,y(n)}) - \nabla f(X^x)\,,\, n(X^x - X^{y(n)})\right\rangle + \left\langle \nabla f(X^x)\,,\, n(X^x - X^{y(n)}) - \zeta^x[\cdot,i]\right\rangle\right]\right| \\
\leqslant\ &\mathbb{E}\left[\|\nabla f(Z^{x,y(n)}) - \nabla f(X^x)\|^2\right]^{1/2} \mathbb{E}\left[n^2\|X^x - X^{y(n)}\|^2\right]^{1/2} &(71) \\
&+\mathbb{E}\left[\|\nabla f(X^x)\|^2\right]^{1/2} \mathbb{E}\left[\|n(X^x - X^{y(n)}) - \zeta^x[\cdot,i]\|^2\right]^{1/2}. &(72)
\end{aligned}$$

Consider the first expectation in (71). We have

$$\begin{aligned}
\sup_n \|\nabla f(Z^{x,y(n)}) - \nabla f(X^x)\| &\leqslant \sup_n \|\nabla f(Z^{x,y(n)})\| + \|\nabla f(X^x)\| \\
&\leqslant \sup_n c(1+\|Z^{x,y(n)}\|^{2p}) + c(1+\|X^x\|^{2p}) \\
&\leqslant c\sup_n \left(1+2^{2p-1}\|X^{y(n)}-X^x\|^{2p} + 2^{2p-1}\|X^x\|^{2p}\right) + c(1+\|X^x\|^{2p}) \\
&\leqslant c\left(1+2^{2p-1}e^{-2pK(t-s)} + 2^{2p-1}\|X^x\|^{2p}\right) + c(1+\|X^x\|^{2p}), \qquad (73)
\end{aligned}$$

where the second inequality uses $\|\nabla f(x)\| \leqslant c(1+\|x\|^{2p})$, the third uses $\|Z^{x,y(n)} - X^x\| \leqslant \|X^{y(n)} - X^x\|$ and the fourth uses Lemma 13. The quantity on the right of the inequality in (73) has finite expectation by Lemma 14. This observation combined with the facts that $Z^{x,y(n)} \to X^x$ a.s. by Lemma 13 and ∇f is continuous, yield via the dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}\left[\|\nabla f(Z^{x,y(n)}) - \nabla f(X^x)\|^2\right]^{1/2} = 0. \qquad (74)$$

For the second expectation in (71), by Lemma 13,

$$\sup_n n^2\|X^x - X^{y(n)}\|^2 \leqslant \sup_n e^{-2K(t-s)} n^2 \|x-y(n)\|^2 = e^{-2K(t-s)},$$

hence

$$\sup_n \mathbb{E}\left[n^2\|X^x - X^{y(n)}\|^2\right]^{1/2} < +\infty. \qquad (75)$$

For the first expectation in (72), again using $\|\nabla f(x)\| \leqslant c(1+\|x\|^{2p})$ and Lemma 14 gives

$$\mathbb{E}\left[\|\nabla f(X^x)\|^2\right]^{1/2} < +\infty. \qquad (76)$$

For the second expectation in (72), Proposition 28 implies

$$\lim_n \mathbb{E}\left[\|n(X^x - X^{y(n)}) - \zeta^x[\cdot,i]\|^2\right]^{1/2} = 0. \qquad (77)$$

Combining (74)-(77) and (71)-(72) establishes (69).

To complete the proof of part 2), it remains to establish the continuity properties. Firstly for the continuity in x, (69) and Cauchy-Schwartz give for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\left|\frac{\partial P_{s,t} f}{\partial x_i}(x) - \frac{\partial P_{s,t} f}{\partial x_i}(y)\right| \\
&\leqslant \mathbb{E}\left[\|\nabla f(X^x) - \nabla f(X^y)\|^2\right]^{1/2} \mathbb{E}\left[\|\zeta^x\|^2\right]^{1/2} + \mathbb{E}\left[\|\nabla f(X^y)\|^2\right]^{1/2} \mathbb{E}\left[\|\zeta^x - \zeta^y\|^2\right]^{1/2}.
\end{aligned}$$

The first expectation converges to zero as $x \to y$ by very similar arguments used above to show (74). The second expectation is finite by (75) and (67). The third expectation converges to $\mathbb{E}\left[\|\nabla f(X^y)\|^2\right]^{1/2}$ using a dominated convergence argument similar to that above and the limit is finite by (78). The fourth expectation converges to zero as $y \to x$ because ζ^x is mean-square continuous in x according to Proposition 28.

Let us next check the continuity in t of $\frac{\partial P_{s,t} f}{\partial x_i}$. Consider (69) and note that $X^x_{s,t}$ and $\zeta^x_{s,t}$ are continuous in t, almost surely. Then due to the almost sure and uniform in t bound on $\|\zeta^x_{s,t}\|_{\text{H.S.}}$ from part 1), the assumption $f \in C^p_2(\mathbb{R}^d)$ and (62), the descired continuity follows by dominated convergence. The continuity in s follows very similar arguments. This completes the proof of part 2). $\square$

Proof of Lemma 30. Throughout the proof, c is a finite constant whose value may change on each appearance.

For part 1),

$$\begin{aligned}
\|\eta^x_{s,t}[\cdot,j,k]\|^2 &\leqslant \frac{2}{\epsilon^2}\sum_{i=1}^d\left(\int_s^t \left\langle D^2F^x_{s,u}[\cdot,\cdot,i]\circ\zeta^x_{s,u}[\cdot,k]\,,\,\zeta^x_{s,u}[\cdot,j]\right\rangle \mathrm{d}u\right)^2 \\
&\quad+\frac{2}{\epsilon^2}\sum_{i=1}^d\left(\int_s^t \left\langle DF^x_{s,u}[\cdot,i]\,,\,\eta^x_{s,u}[\cdot,j,k]\right\rangle \mathrm{d}u\right)^2 \\
&\leqslant \frac{2}{\epsilon^2}\sum_{i=1}^d\left(\int_s^t \|D^2F^x_{s,u}[\cdot,\cdot,i]\|_{\text{H.S.}}\|\zeta^x_{s,u}[\cdot,k]\|\|\zeta^x_{s,u}[\cdot,j]\|\mathrm{d}u\right)^2 \\
&\quad+\frac{2}{\epsilon^2}\sum_{i=1}^d\left(\int_s^t \|DF^x_{s,u}[\cdot,i]\|\|\eta^x_{s,u}[\cdot,j,k]\|\mathrm{d}u\right)^2 \\
&\leqslant \frac{2}{\epsilon^2}(t-s)\int_s^t \|D^2F^x_{s,u}\|^2_{\text{H.S.}}\|\zeta^x_{s,u}[\cdot,k]\|^2\|\zeta^x_{s,u}[\cdot,j]\|^2\mathrm{d}u \\
&+ \frac{2}{\epsilon^2}(t-s)\int_s^t \|DF^x_{s,u}\|^2_{\text{H.S.}}\|\eta^x_{s,u}[\cdot,j,k]\|^2\mathrm{d}u \\
&\leqslant \beta_1+(t-s)\beta_2\int_s^t\|\eta^x_{s,u}[\cdot,j,k]\|^2\mathrm{d}u,
\end{aligned}$$

where the first inequality uses the fact that $\|a+b\|^2\leqslant 2(\|a\|^2+\|b\|^2)$; the second inequality uses Cauchy-Schwartz and the fact for a matrix A and vector b, $\|A\circ b\|\leqslant\sup_{v\neq 0}\frac{\|Av\|}{\|v\|}\|b\|\leqslant\|A\|_{\text{H.S.}}\|b\|$; the third inequality uses Jensen's inequality; the final inequality uses (A6), (A2) and part 1) of Lemma 29, and here β_1,β_2 are finite constants independent of x,j,k,s,t. Gronwall's lemma then gives

$$\|\eta^x_{s,t}[\cdot,j,k]\|^2\leqslant\beta_1\exp[\beta_2(t-s)^2],$$

which completes the proof of part 1) of the lemma.

For part 2), we de-clutter notation as in the proof of Lemma 29 and write

$$X^x\equiv X^x_{s,t},\qquad \zeta^x\equiv\zeta^x_{s,t},\qquad \eta^x\equiv\eta^x_{s,t}.$$

Using (69), we have:

$$\begin{aligned}
&\frac{\partial}{\partial x_i}P_{s,t}f(x)-\frac{\partial}{\partial x_i}P_{s,t}f(y)\\
&=\mathbb{E}\left[\langle\nabla f(X^x)\,,\,\zeta^x[\cdot,i]\rangle\right]-\mathbb{E}\left[\langle\nabla f(X^y)\,,\,\zeta^y[\cdot,i]\rangle\right]\\
&=\mathbb{E}\left[\langle\nabla f(X^x)-\nabla f(X^y)\,,\,\zeta^x[\cdot,i]\rangle\right]+\mathbb{E}\left[\langle\nabla f(X^y)\,,\,\zeta^x[\cdot,i]-\zeta^y[\cdot,i]\rangle\right].
\end{aligned}$$

Therefore to prove the identity in part 2), with $y(n):=x+n^{-1}e_j$ it is sufficient to establish

$$\lim_{n\to\infty}n\mathbb{E}\left[\left\langle\nabla f(X^x)-\nabla f(X^{y(n)})\,,\,\zeta^x[\cdot,i]\right\rangle\right]=\mathbb{E}\left[\left\langle\nabla^{(2)}f(X^x)\circ\zeta^x[\cdot,j]\,,\,\zeta^x[\cdot,i]\right\rangle\right]\tag{78}$$

and

$$\lim_{n\to\infty}n\mathbb{E}\left[\left\langle\nabla f(X^{y(n)})\,,\,\zeta^x[\cdot,i]-\zeta^{y(n)}[\cdot,i]\right\rangle\right]=\mathbb{E}\left[\langle\nabla f(X^x)\,,\,\eta^x[\cdot,i,j]\rangle\right].\tag{79}$$

Using the mean value theorem for vector-valued functions we have:

$$\nabla f(X^x)-\nabla f(X^{y(n)})=\left(\int_0^1\nabla^{(2)}f(X^{y(n)}+u(X^x-X^{y(n)}))du\right)\circ(X^x-X^{y(n)}),$$

where the integral is element-wise. Therefore in terms of the matrices

$$\begin{aligned}
A^{x,y(n)} &:= \nabla^{(2)}f(X^x)-\int_0^1\nabla^{(2)}f(X^{y(n)}+u(X^x-X^{y(n)}))\mathrm{d}u\\
B^{x,y(n)} &:= \int_0^1\nabla^{(2)}f(X^{y(n)}+u(X^x-X^{y(n)}))\mathrm{d}u,
\end{aligned}$$

where the integrals are elemnent-wise, we have

$$
\begin{aligned}
&\left\langle \nabla^{(2)} f(X^x) \circ \zeta^x[\cdot, j]\,,\, \zeta^x[\cdot, i] \right\rangle - n \left\langle \nabla f(X^x) - \nabla f(X^{y(n)})\,,\, \zeta^x[\cdot, i] \right\rangle \\
&= \left\langle \nabla^{(2)} f(X^x) \circ \zeta^x[\cdot, j] - n\left(\nabla f(X^x) - \nabla f(X^{y(n)})\right)\,,\, \zeta^x[\cdot, i] \right\rangle \\
&= \left\langle \left\{ \nabla^{(2)} f(X^x) - \int_0^1 \nabla^{(2)} f(X^{y(n)} + u(X^x - X^{y(n)}))du \right\} \circ \zeta^x[\cdot, j]\,,\, \zeta^x[\cdot, i] \right\rangle \\
&+ \left\langle \left(\int_0^1 \nabla^{(2)} f(X^{y(n)} + u(X^x - X^{y(n)}))du \right) \circ \left\{ \zeta^x[\cdot, j] - n(X^x - X^{y(n)}) \right\}\,,\, \zeta^x[\cdot, i] \right\rangle \\
&\equiv \left\langle A^{x,y(n)} \circ \zeta^x[\cdot, j]\,,\, \zeta^x[\cdot, i] \right\rangle + \left\langle B^{x,y(n)} \circ \left\{ \zeta^x[\cdot, j] - n(X^x - X^{y(n)}) \right\}\,,\, \zeta^x[\cdot, i] \right\rangle.
\end{aligned}
$$

Let us apply dominated convergence to show that

$$
\lim_n \mathbb{E}\left[\left| \left\langle A^{x,y(n)} \circ \zeta^x[\cdot, j]\,,\, \zeta^x[\cdot, i] \right\rangle \right| \right] = 0. \tag{80}
$$

To this end, first note that

$$
\left| \left\langle A^{x,y(n)} \circ \zeta^x_{s,t}[\cdot, j]\,,\, \zeta^x[\cdot, i] \right\rangle \right| \quad \leqslant \quad \|A^{x,y(n)}\|_{\text{H.S.}} \|\zeta^x[\cdot, j]\| \|\zeta^x[\cdot, i]\| \leqslant c \|A^{x,y(n)}\|_{\text{H.S.}},
$$

where c is a finite constant given by part 1) of Lemma 29. Also, by Lemma 13, $X^{y(n)} \to X^x$ a.s., and $\nabla^{(2)} f$ is continuous, hence $\|A^{x,y(n)}\|_{\text{H.S.}} \to 0$ a.s. Also, again using Lemma 13,

$$
\begin{aligned}
\left| A^{x,y(n)}[i,j] \right| \quad &= \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X^x) + \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(X^{y(n)} + u(X^x - X^{y(n)}))\mathrm{d}u \right| \\
&\leqslant \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X^x) \right| + \int_0^1 \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(X^{y(n)} + u(X^x - X^{y(n)})) \right| \mathrm{d}u \\
&\leqslant \quad c(1 + \|X^x\|^{2p}) + c \int_0^1 1 + \|X^{y(n)} + u(X^x - X^{y(n)})\|^{2p} \mathrm{d}u \\
&\leqslant \quad c(1 + \|X^x\|^{2p}) + c \int_0^1 1 + 2^{2p-1}\|X^x\|^{2p} + 2^{2p-1}\|X^x - X^{y(n)}\|^{2p} \mathrm{d}u \\
&\leqslant \quad c(2 + (1 + 2^{2p-1})\|X^x\|^{2p} + 2^{2p-1}).
\end{aligned}
$$

Therefore using Lemma 14, $\mathbb{E}[\sup_{n \geqslant 1} \|A^{x,y(n)}\|_{\text{H.S.}}] < +\infty$, so indeed (80) holds.

Similarly let us now show that

$$
\lim_n \mathbb{E}\left[\left| \left\langle B^{x,y(n)} \circ \left\{ \zeta^x[\cdot, j] - n(X^x - X^{y(n)}) \right\}\,,\, \zeta^x[\cdot, i] \right\rangle \right| \right] = 0. \tag{81}
$$

We have for a finite constant c given by part 1) of Lemma 29,

$$
\begin{aligned}
&\left| \left\langle B^{x,y(n)} \cdot \left\{ \zeta^x_{s,t}[\cdot, j] - n(X^x - X^{y(n)}) \right\}\,,\, \zeta^x[\cdot, i] \right\rangle \right| \\
&\leqslant \left\| B^{x,y(n)} \circ \left\{ \zeta^x_{s,t}[\cdot, j] - n(X^x - X^{y(n)}) \right\} \right\| \|\zeta^x[\cdot, i]\| \\
&\leqslant \|B^{x,y(n)}\|_{\text{H.S.}} \|\zeta^x_{s,t}[\cdot, j] - n(X^x - X^{y(n)})\| c.
\end{aligned}
$$

By very similar arguments used to those used above in bounding $\left|A^{x,y(n)}[i,j]\right|$,

$$
\left| B^{x,y(n)}[i,j] \right| \quad \leqslant \quad c(1 + 2^{2p-1}\|X^x\|^{2p} + 2^{2p-1}),
$$

and therefore by Cauchy-Schwartz,

$$
\begin{aligned}
&\mathbb{E}\left[\left| \left\langle B^{x,y(n)} \cdot \left\{ \zeta^x_{s,t}[\cdot, j] - n(X^x - X^{y(n)}) \right\}\,,\, \zeta^x[\cdot, i] \right\rangle \right| \right] \\
&\leqslant c\mathbb{E}\left[(1 + 2^{2p-1}\|X^x\|^{2p} + 2^{2p-1})^2 \right]^{1/2} \mathbb{E}\left[\|\zeta^x_{s,t}[\cdot, j] - n(X^x - X^{y(n)})\|^2 \right]^{1/2},
\end{aligned}
$$

the first expectation is finite by Lemma 14 and the second converges to zero by Proposition 28. Therefore indeed (81) holds which together with (80) establishes (78).

Our next task is to prove (79). Using Cauchy-Schwartz,

$$
\begin{aligned}
&\left|\mathbb{E}\left[\langle\nabla f(X^x)\,,\,\eta^x[\cdot,i,j]\rangle - \left\langle\nabla f(X^{y(n)})\,,\,n(\zeta^x[\cdot,i]-\zeta^{y(n)}[\cdot,i])\right\rangle\right]\right| \\
&= \mathbb{E}\left[\left|\left\langle\nabla f(X^x)-\nabla f(X^{y(n)})\,,\,\eta^x[\cdot,i,j]\right\rangle\right|\right] + \mathbb{E}\left[\left|\left\langle\nabla f(X^{y(n)})\,,\,\eta^x[\cdot,i,j]-n(\zeta^x[\cdot,i]-\zeta^{y(n)}[\cdot,i])\right\rangle\right|\right] \\
&\leqslant \mathbb{E}\left[\|\nabla f(X^x)-\nabla f(X^{y(n)})\|^2\right]^{1/2}\mathbb{E}\left[\|\eta^x[\cdot,i,j]\|^2\right]^{1/2} \\
&+\mathbb{E}\left[\|\nabla f(X^{y(n)})\|^2\right]^{1/2}\mathbb{E}\left[\|\eta^x[\cdot,i,j]-n(\zeta^x[\cdot,i]-\zeta^{y(n)}[\cdot,i])\|^2\right]^{1/2}.
\end{aligned}
$$

The first expectation converges to zero as $n\to\infty$ by arguments very similar to those used to prove (74). The second expectation is finite, since we have already established that $\|\eta^x[\cdot,i,j]\|$ is bounded by a finite constant, a.s. By yet another dominated convergence argument, the third expectation converges to $\mathbb{E}\left[\|\nabla f(X^{y(n)})\|^2\right]^{1/2}$, which is finite by (76). The fourth expectation converges to zero by Proposition 28. The proof of (70) is complete.

To complete the proof of the Lemma it remains to verify that $\nabla^{(2)}P_{s,t}f(x)$ is continuous in x, s and t. From (70) we consider:

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle\nabla^{(2)}f(X^x)\circ\zeta^x[\cdot,j]\,,\,\zeta^x[\cdot,i]\right\rangle\right]-\mathbb{E}\left[\left\langle\nabla^{(2)}f(X^y)\circ\zeta^y[\cdot,j]\,,\,\zeta^y[\cdot,i]\right\rangle\right] \\
&=\mathbb{E}\left[\left\langle\left\{\nabla^{(2)}f(X^x)-\nabla^{(2)}f(X^y)\right\}\circ\zeta^x[\cdot,j]\,,\,\zeta^x[\cdot,i]\right\rangle\right] \\
&+\mathbb{E}\left[\left\langle\nabla^{(2)}f(X^y)\circ\{\zeta^x[\cdot,j]-\zeta^y[\cdot,j]\}\,,\,\zeta^x[\cdot,i]\right\rangle\right] \\
&+\mathbb{E}\left[\left\langle\nabla^{(2)}f(X^y)\circ\zeta^y[\cdot,j]\,,\,\zeta^x[\cdot,i]-\zeta^y[\cdot,i]\right\rangle\right].
\end{aligned}
$$

All three of these expectations converge to zero as $y\to x$, by arguments involving dominated convergence and the mean-square continuity of $\zeta^x_{s,t}$ asserted in Proposition 28. The details are omitted. Similary

$$
\begin{aligned}
&\mathbb{E}\left[\langle\nabla f(X^x)\,,\,\eta^x[\cdot,i,j]\rangle\right]-\mathbb{E}\left[\langle\nabla f(X^y)\,,\,\eta^y[\cdot,i,j]\rangle\right] \\
&=\mathbb{E}\left[\langle\nabla f(X^x)-\nabla f(X^y)\,,\,\eta^x[\cdot,i,j]\rangle\right]+\mathbb{E}\left[\langle\nabla f(X^y)\,,\,\eta^x[\cdot,i,j]-\eta^y[\cdot,i,j]\rangle\right]
\end{aligned}
$$

converges to zero as $y\to x$ again using dominated convergence, and the mean-square continuity in x of η^x asserted in Proposition 28. The continuity of $\frac{\partial^2 P_{s,t}f}{\partial x_i\partial x_j}$ in s and t follows from very similar arguments to those used to prove the continuity of $\frac{\partial P_{s,t}}{\partial x_i}$ in Lemma 29. $\square$

Proof of Proposition 15. Lemmas 29 and 30 together establish that for $q=1,2$, if $f\in C^p_q(\mathbb{R}^d)$ then $P_{s,t}f$ is q-times continuously differentiable in x, and by (46), $P_{s,t}f\in C^p_0(\mathbb{R}^d)$. To complete the proof of (47), it remains to obtain suitable bounds on $\|\nabla P_{s,t}f\|$ and $\|\nabla^{(2)}P_{s,t}f\|^2_{\mathrm{H.S.}}$. Using (69), (70), the almost sure bounds on $\|\zeta^x_{s,t}\|_{\mathrm{H.S}}$, $\|\eta^x_{s,t}\|_{\mathrm{H.S}}$, and Lemma 14, we have for some finite constant c depending only on f,

$$
\begin{aligned}
\|\nabla P_{s,t}f(x)\|^2 &= \sum_{i=1}^d\mathbb{E}\left[\langle\nabla f(X^x_{s,t})\,,\,\zeta^x_{s,t}[\cdot,i]\rangle\right]^2 \\
&\leqslant \sum_{i=1}^d\mathbb{E}\left[\|\nabla f(X^x_{s,t})\|\|\zeta^x_{s,t}[\cdot,i]\|\right]^2 \\
&\leqslant dc^2c_1^2\left(1+\mathbb{E}\left[\|X^x_{s,t}\|^{2p}\right]\right)^2 \\
&\leqslant dc^2c_1^2\alpha_p^2(1+\|x\|^{2p})^2
\end{aligned}
\tag{82}
$$

and similarly

$$
\begin{aligned}
\|\nabla^{(2)} P_{s,t} f(x)\|_{\mathrm{H.S.}}^2 &= \sum_{i,j=1}^{d} \left\{ \mathbb{E}\left[\left\langle \nabla^{(2)} f(X_{s,t}^x) \circ \zeta_{s,t}^x[\cdot, j]\,,\, \zeta_{s,t}^x[\cdot, i] \right\rangle \right] + \mathbb{E}\left[\left\langle \nabla f(X_{s,t}^x)\,,\, \eta_{s,t}^x[\cdot, i, j] \right\rangle \right] \right\}^2 \\
&\leqslant \sum_{i,j=1}^{d} 2\mathbb{E}\left[\|\nabla^{(2)} f(X_{s,t}^x)\|_{\mathrm{H.S}} \|\zeta_{s,t}^x[\cdot, j]\| \|\zeta_{s,t}^x[\cdot, j]\| \right]^2 + 2\mathbb{E}\left[\|\nabla f(X_{s,t}^x)\| \|\eta_{s,t}^x[\cdot, i, j]\| \right]^2 \\
&\leqslant 2d^2 c_1^4 c^2 \left(1 + \mathbb{E}\left[\|X_{s,t}^x\|^{2p}\right]\right)^2 + 2d^2 c_2^2 c^2 \left(1 + \mathbb{E}\left[\|X_{s,t}^x\|^{2p}\right]\right)^2 \\
&\leqslant 2d^2 (c_1^4 + c_2^2) c^2 \alpha_p^2 (1 + \|x\|^{2p})^2. \qquad (83)
\end{aligned}
$$

The proof of (47) is then complete.

Now consider the first inclusion in (48). Observe that since $f \in C_{1,2}^p([0,1] \times \mathbb{R}^d)$ and (A5) holds, $|\partial_t f_t(x)| + |\mathcal{L}_t f_t(x)|$ is continuous in t and x, and there exists a finite constant c such that

$$
\begin{aligned}
|\partial_t f_t(x)| + |\mathcal{L}_t f_t(x)| &\leqslant |\partial_t f_t(x)| + \epsilon^{-1} \|\nabla U_t(x)\| \|\nabla f_t(x)\| + \epsilon^{-1} |\Delta f_t(x)| \qquad (84) \\
&\leqslant c(1 + \|x\|^{2p}) \left[1 + \epsilon^{-1} \|\nabla U_t(x)\| + d\epsilon^{-1}\right].
\end{aligned}
$$

The proof of (48) is then completed by noting (A3).

For the remaining inclusion of (48), note that $\mathcal{L}_s P_{s,t} f_t(x)$ is continuous in s and x by (A5) and the second parts of Lemmas 29 and 30. Also

$$
|\mathcal{L}_s P_{s,t} f_t(x)| \leqslant \epsilon^{-1} \|\nabla U_s(x)\| \|\nabla P_{s,t} f_t(x)\| + \epsilon^{-1} |\Delta P_{s,t} f_t(x)|,
$$

so the proof is complete upon again noting (A3) and the fact that the constants in (82), (83) are independent of s. □

4.3 Proof and supporting results for Proposition 16

Proof of Proposition 16. Fix $s \in [0,1]$ and $x \in \mathbb{R}^d$. Define $T_m := \inf\{t \geqslant s : \|X_{s,t}^x\| > m\}$, the dependence of T_m on x and s is not shown in the notation. By non-explosivity of the process, $T_m \to \infty$, a.s. Write $\mathcal{L} f(t,x) \equiv \partial_t f(x) + \mathcal{L}_t f_t(x)$.

By Dynkin's formula [23, Lem. 3.2, p.73],

$$
\mathbb{E}\left[f(T_m \wedge t, X_{s, T_m \wedge t}^x) \right] = f(s,x) + \mathbb{E}\left[\int_s^{T_m \wedge t} \mathcal{L} f(u, X_{x,u}^x) \mathrm{d}u \right], \qquad (85)
$$

and therefore using equation (48) of Proposition 15,

$$
\begin{aligned}
\sup_m |f(T_m \wedge t, X_{s, T_m \wedge t}^x)| &\leqslant |f(s,x)| + \sup_m \int_s^{T_m \wedge t} |\mathcal{L} f(u, X_{x,u}^x)| \mathrm{d}u \\
&\leqslant |f(s,x)| + \int_s^t c(1 + \|X_{s,u}^x\|^{2p+1}) \mathrm{d}u. \qquad (86)
\end{aligned}
$$

The expected value of (86) is finite due to equation (46) of Lemma 14 and Fubini, so combined with the fact that $f(T_m \wedge t, X_{s, T_m \wedge t}^x) \to f(t, X_{s,t}^x)$, a.s., dominated convergence may be applied to (85) and Fubini applied once more to give:

$$
\mathbb{E}[f(t, X_{s,t}^x)] = f(s,x) + \int_s^t \mathbb{E}\left[\mathcal{L} f(u, X_{s,u}^x) \right] \mathrm{d}u.
$$

Integrating with respect to ν and using (48), (46) and the assumption $\nu \in \mathcal{P}^{p+1/2}(\mathbb{R}^d)$ to validate changing the order of integration we obtain

$$
\int_{\mathbb{R}^d} \mathbb{E}[f(t, X_{s,t}^x)] \nu(\mathrm{d}x) = \int_{\mathbb{R}^d} f(s,x) \nu(\mathrm{d}x) + \int_s^t \int_{\mathbb{R}^d} \mathbb{E}\left[\mathcal{L} f(u, X_{s,u}^x) \right] \nu(\mathrm{d}x) \mathrm{d}u. \qquad (87)
$$

By Lemma 27, $\int_{\mathbb{R}^d} \mathbb{E}\left[\mathcal{L}f(u, X^x_{s,u})\right] \nu(\mathrm{d}x)$ is continuous in u, and so (87) is differentiable in t and (49) holds.

Fix t and write $g_s(x) := P_{s,t}f(x) = \mathbb{E}[f(X^x_{s,t})]$, and note that $g_s(x) = P_{s,s+\delta}P_{s+\delta,t}f(x) = \mathbb{E}[g_{s+\delta}(X^x_{s,s+\delta})]$. Observe that by (47) for any s, $x \mapsto g_s(x) \in C^p_2(\mathbb{R}^d)$, and also using (A3) and noting that the constants in (82) and (83) do not depend on s. there exists a finite constant c such that

$$\sup_\tau |\Delta g_\tau(x)| \vee \sup_\tau \|\nabla g_\tau(x)\| \vee \sup_\tau \|\nabla U_\tau(x)\| \leqslant c(1 + \|x\|^{2p}), \quad \forall x. \tag{88}$$

Therefore by an application of Ito's formula, (46) and Fubini, for any $\delta > 0$,

$$\begin{aligned} g_s(x) - g_{s+\delta}(x) &= \mathbb{E}[g_{s+\delta}(X^x_{s,s+\delta})] - g_{s+\delta}(x) \\ &= \int_s^{s+\delta} \mathbb{E}\left[-\epsilon^{-1}\left\langle \nabla g_{s+\delta}(X^x_{s,u}), \nabla U_u(X^x_{s,u})\right\rangle + \epsilon^{-1}\Delta g_{s+\delta}(X^x_{s,u})\right] \mathrm{d}u \\ &= \mathbb{E}\left[-\epsilon^{-1}\left\langle \nabla g_{s+\delta}(X^x_{s,\tau}), \nabla U_\tau(X^x_{s,\tau})\right\rangle + \epsilon^{-1}\Delta g_{s+\delta}(X^x_{s,\tau})\right] \delta, \end{aligned} \tag{89}$$

where the final equality is valid for some τ in the interval $(s, s+\delta)$ since the expectation in (89), which is equal to $P_{s,u}\mathcal{L}_u g_{s+\delta}(x)$, depends continuously on u due to (48) and the continuity part of Lemma 27. Then using (88), (46), Lemma 27 and dominated convergence in order to interchange limits and expectation,

$$\lim_{\delta \to 0} \frac{g_s(x) - g_{s+\delta}(x)}{\delta} = \mathcal{L}_s g_s(x).$$

A similar argument applied to $[g_{s-\delta}(x) - g_s(x)]\delta^{-1}$ gives the same limit, which establishes (50).

It remains to check that the map $(s,x) \mapsto P_{s,t}f_t(x)$ is a member of $C^{p+1/2}_{1,2}([0,1] \times \mathbb{R}^d)$. By (46), $\sup_{s,x} |P_{s,t}f_t(x)|/(1 + \|x\|^{2p}) < +\infty$; we have already proved $P_{s,t}f_t(x)$ is differentiable in s and its derivative is $-\mathcal{L}_s P_{s,t}f_t(x)$; by Proposition 15 $\mathcal{L}_s P_{s,t}f_t(x)$ is continous in s and $\sup_{s,x} |\mathcal{L}_s P_{s,t}f_t(x)|/(1+\|x\|^{2p+1}) < +\infty$; by (47), for any s, $P_{s,t}f_t \in C^p_2(\mathbb{R}^d)$, and the proof is completed upon noting that the constants in (82) and (83) do not depend on s. □

5 Quantitative CLT bound for the diffusion skeleton

5.1 Main result

We assume throughout section 5 that for $s \in [0,1]$ and $\epsilon > 0$ $\mu_s^\epsilon f_s = 0$ (and note that f_s therefore implicitely depends on ϵ) and we let $\bar{f}_s = f_s - \pi_s f_s$. Let$(B_t)_{t \in \mathbb{R}_+}$ be a $d-$dimensional Brownian motion. For any $\epsilon > 0$, we define $\left(X_t^\epsilon\right)_{t \in [0,1]}$ as the continuous solution for $t \in [0,1]$ of

$$X_t^\epsilon = X_0^\epsilon - \epsilon^{-1} \int_0^t \nabla U_u(X_u^\epsilon) \mathrm{d}u + \sqrt{2\epsilon^{-1}} \int_0^t \mathrm{d}B_u, \tag{90}$$

with $X_0^\epsilon =: X_0$ $\mathcal{F}_0-$measurable and of distribution μ_0. One may be interested in the distributional limiting behaviour as $\epsilon \to 0$ of

$$\epsilon^{-1/2} S_\epsilon = \epsilon^{-1/2} \int_0^1 \bar{f}_t(X_t^\epsilon) \mathrm{d}t,$$

and it is expected that a central limit theorem (CLT) may hold. We do not focus on this here, but rather investigate the following related problem. Define, for any $h \in (0,1)$, quantities resulting from a Riemann sum approximation of the integral above,

$$\epsilon^{-1/2} S_{\epsilon,h} := \epsilon^{-1/2} h \sum_{i=0}^{n-1} \bar{f}_{ih}(X_{ih}^\epsilon).$$

where $n := \lfloor 1/h \rfloor$ (note that $n \geqslant 1$ by assumption). The aims of this section are to characterize $\lim_{\epsilon \to 0} \operatorname{var}\left[\epsilon^{-1/2} S_{\epsilon,h(\epsilon)}\right]$ and the limiting distributional behaviour of $\epsilon^{-1/2} S_{\epsilon,h(\epsilon)}$ as $\epsilon \to 0$, for various choices of $h(\cdot) : \mathbb{R}_+ \to (0,1)$. Note that in order to alleviate notation below we may use h for $h(\epsilon)$ when no confusion is possible.

In order to present the main result of this section we introduce quantities related to the following family of time homogeneous and stationary processes $\left(Y_t^{s,\epsilon}\right)_{s,t \in [0,1], \epsilon > 0}$. Let for any $s \in [0,1]$, $\epsilon > 0$, $t \in \mathbb{R}_+$,

$$Y_t^{s,\epsilon} = Y_0^{s,\epsilon} - \epsilon^{-1} \int_0^t \nabla U_s(Y_u^{s,\epsilon}) \mathrm{d}u + \sqrt{2\epsilon^{-1}} \int_0^t \mathrm{d}B_u$$

with $Y_0^{s,\epsilon} =: Y_0^s$ $\mathcal{F}_0-$measurable of distribution π_s. We naturally use $\mathbb{P}\left[\,\cdot\,\right]$ and $\mathbb{E}\left[\,\cdot\,\right]$ for the laws and expectations of both $\left(X_t^\epsilon\right)_{t \in [0,1], \epsilon > 0}$ and $\left(Y_t^{s,\epsilon}\right)_{(s,t) \in [0,1] \times \mathbb{R}_+, \epsilon > 0}$. For $s \in [0,1]$ we let $L_2(\pi_s)$ be the set of real valued and π_s-square integrable functions on $\mathbb{R}^d$. For any $s, t \in [0,1]$, $f \in L^2(\pi_s)$, $\epsilon > 0$ and $x \in \mathbb{R}^d$ we let $Q_t^{s,\epsilon} f(x) := \mathbb{E}\left[f(Y_t^{s,\epsilon}) \mid Y_0^s = x\right]$ and $Q_t^s f(x) := Q_t^{s,1} f(x)$ and $P_{s,t}^\epsilon f(x) := \mathbb{E}\left[f(X_t^\epsilon) \mid X_s = x\right]$. Standard results on stationary reversible Markov processes and Markov chains, together with our geometric ergodicity assumptions ensure that the following limits exist and are finite for $f_s \in L^2(\pi_s)$,

$$\varsigma_0(s) := \lim_{\epsilon \to 0} \operatorname{var}\left[\epsilon^{-1/2} \int_0^1 \bar{f}_s(Y_t^{s,\epsilon}) \mathrm{d}t\right] \text{ and } \varsigma_\ell(s) := \lim_{\epsilon \to 0} \operatorname{var}\left[\epsilon^{-1/2} h(\epsilon) \sum_{i=0}^{n-1} \bar{f}_s(Y_{ih(\epsilon)}^{s,\epsilon})\right] \text{ for } \ell = h(\epsilon)\epsilon^{-1} > 0,$$

where $\operatorname{var}\left[\,\cdot\,\right]$ is the variance operator associated with $\mathbb{E}\left[\,\cdot\,\right]$. Note the broad use we make throughout of ℓ to refer to scenarios and not just a numerical value. It is well known that the following upper bounds, in terms of either spectral gap or K in (A4), hold

$$\varsigma_\ell(s) \leqslant 2\operatorname{var}_{\pi_s}\left(f_s\right) \cdot \begin{cases} \ell \mathrm{Gap}_R\left(Q_\ell^s\right)^{-1} \leqslant \left[(1 - \exp(-K\ell))/\ell\right]^{-1} & \text{for } \ell > 0 \\ \mathrm{Gap}\left(\mathcal{L}_s\right)^{-1} \leqslant K^{-1} & \text{for } \ell = 0 \end{cases}.$$

The last inequality follows from the fact that from Poincaré's inequality $\operatorname{var}_{\pi_s}\left[f_s\right] \leqslant K^{-1} \mathcal{E}_{\mathcal{L}_s}\left[f_s\right]$ (with $\mathcal{E}_{\mathcal{L}_s}\left[f_s\right] := -\int \bar{f}_s \mathcal{L}_s \bar{f}_s) \mathrm{d}\pi_s)$ and the variational representation of the spectral gap. These spectral gap bounds are classic, and can, for example, be deduced from the spectral representations in Theorem 35. Under our assumptions, for any $\ell \geqslant 0$, $s \mapsto \varsigma_\ell(\cdot), \operatorname{var}_{\pi_s}\left(f_s\right)$ can be shown to be continuous functions (see the proof of Lemma 53, which exploits the results of Lemma 62 and the representation (96) of $\varsigma_\ell(\cdot)$), and

$$\sigma_\ell^2 := \int_0^1 \varsigma_\ell(s) \mathrm{d}s \tag{91}$$

is therefore well defined. The results of this section rely on the following assumptions. We consider a sequence of processes as above, indexed by the dimension of the problem d, for which we assume the following.

(**A10**) (Polynomial dependence on dimension)We assume that (A7) holds and that in addition $\sup_{s\in[0,1]} \|\partial_t f_s\|_p$ and $\sup_{s\in[0,1]} 1/\varsigma^{(d)}(s)$ grow at most polynomially in d as $d \to \infty$.

We impose the following dependence of h on .

(**A11**) (Dependence between ϵ and h)

1. for any $\ell > 0$ we set $h(\epsilon) := \ell\epsilon$,
2. for $\ell = 0$ we set $h(\epsilon) = O\big(\epsilon^c\big)$ for some $c > 1$.

We can now formulate our first result. Throughout C is a constant, not dependent on the quantities in assumptions (A1-5), and whose value may change upon each appearance.

Theorem 31. *Let $p \geqslant 1$ and for any $d \in \mathbb{N}$, let $(X_t^\epsilon(d))_{t\in[0,1]}$ be as defined in (90) and $f^{(d)} \in C^p_{1,2}([0,1]\times\mathbb{R}^d)$. Assume that for any $d \in \mathbb{N}$ (A1-5) and (A10) hold. Then for any $\ell \geqslant 0$ there exists $a > 0$ such that with $\epsilon(d) = O(d^{-a})$ and $d \mapsto h(d)$ satisfying (A11), then*

$$\lim_{d\to\infty}\left|\mathrm{var}\left[\epsilon(d)^{-1/2}S_{\epsilon(d),h(d)}\right] - \sigma_\ell^2(d)\right| = 0.$$

Remark 32. It is in principle possible to get quantitative bounds on the rate of convergence above, by agreggation of the various bounds found in our proof. We do not pursue this here due to a lack of space and the limited interest of such bounds in practice.

Remark 33. It should be clear from the proof that under (A1-5) and (A11), the result holds for fixed d as $\epsilon \to 0$, that is

$$\lim_{\epsilon\to 0}\left|\mathrm{var}\left[\epsilon^{-1/2}S_{\epsilon,h(\epsilon)}\right] - \sigma_\ell^2\right| = 0.$$

Remark 34. Note that compared to the results concerned with the mean square error convergence, we have made the additional assumptions that $s \mapsto f_s(x)$ is continuously differentiable for any $x \in \mathbb{R}^d$. This condition is required in order to obtain explicit control on the error in Riemanian sums involved in our calculations, and could be relaxed easily to Hölder continuity, at the expense of additional notation.

As an aside, it is natural to investigate the impact of ℓ on this asymptotic variance σ_ℓ^2. The following result confirms our intuition that the smaller ℓ, the better; the result below can be understood as being a generalisation of [16, Theorem 3.3], an important fact in the area of discrete time Markov chain Monte Carlo methods, concerned with thinning in the context of ergodic averages.

Theorem 35. *For $s \in [0,1]$ and any $f_s \in L^2(\pi_s)$ there exists a non-negative measure ν_s on $\big([0,\infty), \mathcal{B}([0,\infty))\big)$ such that for $\ell > 0$*

$$\varsigma_\ell(s) = \ell\int_0^\infty \frac{1+\exp(-\ell\lambda)}{1-\exp(-\ell\lambda)}\nu_s(\mathrm{d}\lambda),$$

and

$$\varsigma_0(s) = 2\int_0^\infty \lambda^{-1}\nu_s(\mathrm{d}\lambda).$$

Further, for any $s \in [0,1]$, $\ell \mapsto \varsigma_\ell(s)$ is a non-decreasing function on $[0,\infty)$.

Proof. Let for any $s \in [0,1]$ and $f, g \in L_2(\pi_s)$,$\langle f, g\rangle_{\pi_s} := \int fg\mathrm{d}\pi_s$. For $\ell > 0$ the first statement follows from the fact that $-\mathcal{L}_s$ is a positive self-adjoint operator, implying that one can apply the spectral decomposition theorem and establish that ([1, Section 1.7.2 & Appendix A4])

$$\big\langle \bar{f}_s, Q_t^s\bar{f}_s\big\rangle_{\pi_s} = \int_0^\infty \exp\big(-t\lambda\big)\nu_s(\mathrm{d}\lambda),$$

from which one can conclude by noting that, with $\mathrm{cov}[\cdot,\cdot]$ the covariance operator associated with $\mathbb{E}[\cdot]$, for any $\epsilon > 0$

$$\begin{aligned}\mathrm{var}\left[\epsilon^{-1/2}h\sum_{i=0}^{n-1}\bar{f}_s(Y_{ih}^{s,\epsilon})\right] &= \epsilon^{-1}h^2\left(n\mathrm{var}_{\pi_s}[f_s]+2\sum_{k=1}^{n-1}(n-k)\mathrm{cov}\big[f_s(Y_0^{s,\epsilon}),f_s(Y_{kh}^{s,\epsilon})\big]\right)\\ &= \epsilon^{-1}h(nh)\left(\mathrm{var}_{\pi_s}[f_s]+2\sum_{k=1}^{n-1}(1-k/n)\big\langle\bar{f}_s,Q_{kh\epsilon^{-1}}^{s}\bar{f}_s\big\rangle_{\pi_s}\right),\end{aligned}$$

and using standard convergence arguments. The case $\ell = 0$ is naturally standard. For $\lambda \in (0,\infty)$ (we have a positive spectral gap, so all cases are covered) consider the function

$$\varphi_\lambda(\ell) := \ell\frac{1+\exp(-\ell\lambda)}{1-\exp(-\ell\lambda)} = \ell\left(\frac{2}{1-\exp(-\ell\lambda)}-1\right).$$

We show that it is non-decreasing on $(0,\infty)$, as a function of ℓ. We have

$$\begin{aligned}\varphi_\lambda'(\ell) &= \left(\frac{2}{1-\exp(-\ell\lambda)}-1\right)-\ell\frac{2\lambda\exp(-\ell\lambda)}{(1-\exp(-\ell\lambda))^2}\\ &= \frac{(1+\exp(-\ell\lambda))(1-\exp(-\ell\lambda))-2\ell\lambda\exp(-\ell\lambda)}{(1-\exp(-\ell\lambda))^2}.\\ &= \frac{1-\exp(-2\ell\lambda)-2\ell\lambda\exp(-\ell\lambda)}{(1-\exp(-\ell\lambda))^2}.\end{aligned}$$

Consider the function $D(a) := 1-\exp(-2a)-2a\exp(-a)$ and note that its derivative is $D'(a) = 2\exp(-2a)+2(a-1)\exp(-a) = 2\exp(-a)[a-1+\exp(-a)]$. Therefore $D'(a) \geqslant 0$ and since $D(0) = 0$ we deduce $D(a) \geqslant 0$ for $a \geqslant 0$. We therefore conclude that $\varphi_\lambda'(\ell) \geqslant 0$ for $\ell > 0$. Finally we notice that for $\lambda > 0$

$$\lim_{\ell\to 0}\ell\frac{1+\exp(-\ell\lambda)}{1-\exp(-\ell\lambda)} = 2/\lambda$$

and therefore for $\ell > 0$, $\varphi_\lambda(\ell) > 2/\lambda$, from which we conclude. □

Let $\Phi(\cdot)$ be the cumulative distribution function of the standardized normal distribution. The main result of this section is

Theorem 36. *Let $p \geqslant 1$ and for any $d \in \mathbb{N}$, let $(X_t^\epsilon(d))_{t\in[0,1]}$ be as defined in (90) and $f^{(d)} \in C^p_{1,2}([0,1]\times\mathbb{R}^d)$. Assume that for any $d \in \mathbb{N}$ (A1-5) and (A10) hold. Then for any $\ell \geqslant 0$ there exists $a > 0$ such that with $\epsilon(d) = O(d^{-a})$ and $d \mapsto h(d)$ satisfying (A11), then*

$$\lim_{d\to\infty}\sup_{w\in\mathbb{R}}\left|\mathbb{P}\left[\epsilon(d)^{-1/2}S_{\epsilon(d),h(d)}/\sqrt{\sigma_\ell^2(d)} \leqslant w\right]-\Phi(w)\right| = 0.$$

As we shall see later on, the scenario we are particularly interested in corresponds to the choice $h(d) = o\big(\epsilon(d)^2/d\big)$ or $h = h(\epsilon) = O\big(\epsilon(d)^2/d\big)$ as $d \to \infty$ (or even fixed d and $\epsilon \to 0$), in which case the CLT is inherited by the discretized Langevin process, see Section 7. The proof of the theorem above relies on a martingale approximation and a quantitative bound for the CLT for martingales.

Proof. First we consider the upper bound suggested by Proposition 38. Then we choose $\varepsilon_1(d) = Cd^{-c}$ with $c \in (0,1/2)$ as in Lemma 39 and Lemma 40, $\varepsilon_2(d)$ as in Corollary 48 with, say $r_2 > 1/2$, implying that $\lim_{d\to\infty}\varepsilon_1(d)\varepsilon_2^{-1}(d) = \infty$. The result then follows from Theorem 41. □

5.2 Quantitative Martingale approximation for the CLT

The main result of this section is Proposition 38 which establishes a bound on $\sup_{w\in\mathbb{R}}\big|\mathbb{P}\big[S_{\epsilon,h}/\sqrt{\epsilon\sigma_\ell^2} \leqslant w\big]-\Phi(w)\big|$ in terms of the sum of $\sup_{w\in\mathbb{R}}\big|\mathbb{P}\big[M_\epsilon \leqslant w\big]-\Phi(w)\big|$, where M_ϵ is the last term of a Martingale sequence, and additional negligible terms for which we derive quantitative bounds. We find a quantitative upper bound on $\sup_{w\in\mathbb{R}} \left|\mathbb{P}\left[M_\epsilon \leqslant w\right] - \Phi(w)\right|$ in section 5.3. There are essentially two routes to constructing such an approximation. An approach consists of using solutions to the set of time homogeneous Poisson equations $\tilde{f}_s - Q^s_{h\epsilon^{-1}}\tilde{k}_s = \bar{f}_s$, but we here follow an approach inspired by [34], which consists of treating bias and variance separately by centering f_t around $\mu^\epsilon_t f_t$, and not $\pi_t f_t$. Note that we have also avoided the use of the solutions of the Poisson equation for the continous time processes involved (that is either $\mathcal{L}_s\hat{f}_s = -\bar{f}_s$ or its time inhomogeneous counterpart) as this would have required quantitative bounds on their gradients with respect to x and on their time derivatives. Such bounds are currently not available with sufficient generality [31, 30, 35] to cover our scenario. We introduce

$$B_{\epsilon,h} := h\sum_{k=0}^{n-1} \pi_{kh} f_{kh} - \mu^\epsilon_{kh} f_{kh},$$

and construct our martingale approximation of $S_{\epsilon,h}/\sqrt{\epsilon\sigma^2_\ell}$. Following [34] we introduce for $k \in \{0,\ldots,n-1\}$ and $x \in \mathbb{R}^d$

$$\gamma_{k,\epsilon}(x) : = \sum_{i=k}^{n-1} P^\epsilon_{kh,ih} f_{ih}(x).$$

Remark that for $0 \leqslant k \leqslant n-2$, $\gamma_{k,\epsilon}$ satisfies

$$f_{kh}(x) = \gamma_{k,\epsilon}(x) - P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(x) \tag{92}$$

for any $x \in \mathbb{R}^d$–this can be thought of as a generalization of Poisson's equation. In order to formulate our explicit bounds concisely and in a unified manner we introduce some notation and establish useful identities in Proposition 57. Define for $q > 0$ $V^{(q)}(x) := \|x\|^{2q}$, $\bar{V}^{(q)}(x) := 1+\|x\|^{2q}$, $\bar{V}^{(q)}_t(x) := 1+V^{(q)}_t(x) := 1+\|x-x^\star_t\|^{2q}$ (with notational simplifications $\bar{V}_t := \bar{V}^{(1)}_t$ and $V_t := V^{(1)}_t$ etc.). In addition to what is proposed in Section 1.2, for $f : [0,1]\times\mathbb{R}^d \to \mathbb{R}$ we let $\|\partial_t f\|_p := \sup_{t\in[0,1]}\|\partial_t f_t\|_p$ and $\|\nabla^{(r)} f\|_p := \sup_{t\in[0,1]}\|\nabla^{(r)} f_t\|_p$. We let $|||f|||_p := \|f\|_{\bar{V}^{(p)}} \vee \|\nabla f\|_{\bar{V}^{(p)}} \vee \|\Delta f\|_{\bar{V}^{(p)}}$.

Lemma 37. *Let $p \geqslant 1$ and $f \in C^p_{0,2}([0,1]\times\mathbb{R}^d)$.*

1. *For any $\epsilon, h > 0$ and $k \in \{0,\ldots,n-1\}$, $\gamma_{k,\epsilon} \in C^p_2([0,1]\times\mathbb{R}^d)$ and we have the quantitative bound*

$$\max_{k\in\{0,\ldots,n-1\}}\left\{\left|P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(x)\right| \vee \left|\gamma_{k,\epsilon}(x)\right|\right\} \leqslant C\alpha_p \frac{\|\nabla f\|_p}{1-\exp\left(-K\epsilon^{-1}h\right)}\mu_0\bar{V}^{(p+1/2)}\cdot\bar{V}^{(p+1/2)}(x).$$

2. *$\mathbb{P}-a.s.$ we have*

$$\sum_{k=0}^{n-1} f_{kh}(X^\epsilon_{kh}) = \gamma_{0,\epsilon}(X^\epsilon_0) + \sum_{k=1}^{n-1}\gamma_{k,\epsilon}(X^\epsilon_{kh}) - P^\epsilon_{(k-1)h,kh}\gamma_{k,\epsilon}(X^\epsilon_{(k-1)h}),$$

3. *For $1 \leqslant k \leqslant n-1$ define $\xi_{k,\epsilon} := \left(\gamma_{k,\epsilon}(X^\epsilon_{kh}) - P^\epsilon_{(k-1)h,kh}\gamma_{k,\epsilon}(X^\epsilon_{(k-1)h})\right)$, $\xi_{0,\epsilon} := 0$,*

$$\upsilon(\epsilon) := \epsilon^{-1}h^2\mathrm{var}\left[\sum_{i=0}^{n-1}\xi_{i,\epsilon}\right],$$

and for $0 \leqslant k \leqslant n-1$ and $\epsilon > 0$ such that $\upsilon(\epsilon) > 0$ we let

$$M_{k,\epsilon} := \epsilon^{-1/2}h\sum_{i=0}^{k}\xi_{i,\epsilon}/\sqrt{\upsilon(\epsilon)}.$$

Then $\left(M_{i,\epsilon}, \mathcal{F}_{ih}\right)_{i\in\{0,\ldots,n-1\}}$ is a martingale.

Proof. For notational simplicity we drop ϵ from $P^{\epsilon}_{s,t}$ here. For the first statement we first apply Proposition 15 and then use Lemma 24 in order to obtain the quantitative bound : for any $x \in \mathbb{R}^d$

$$|\delta_x P_{0,t} f_t| \leqslant \alpha_p \|\nabla f_t\|_p W^{(p)}(\delta_x, \pi_0) \exp\big(-K\epsilon^{-1}t\big)$$

and therefore for $k \in \{0, \ldots, n-1\}$, using Lemma 56,

$$\begin{aligned}\big|P^{\epsilon}_{kh,(k+1)h}\gamma_{k+1,\epsilon}(x)\big| \vee \big|\gamma_{k,\epsilon}(x)\big| &\leqslant \alpha_p \frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)} W^{(p)}(\delta_x, \mu_0) \\ &\leqslant C\alpha_p \frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)} \mu_0 \bar{V}^{(p+1/2)} \cdot \bar{V}^{(p+1/2)}(x).\end{aligned}$$

The second statement: from (92) we have for $1 \leqslant k \leqslant n-2$

$$f_{kh}(X^{\epsilon}_{kh}) = \gamma_{k,\epsilon}(X^{\epsilon}_{kh}) - P_{(k-1)h,kh}\gamma_{k,\epsilon}(X^{\epsilon}_{(k-1)h}) + P_{(k-1)h,kh}\gamma_{k,\epsilon}(X^{\epsilon}_{(k-1)h}) - P_{kh,(k+1)h}\gamma_{k+1,\epsilon}(X^{\epsilon}_{kh})$$

and therefore

$$\sum_{k=1}^{n-2} f_{kh}(X^{\epsilon}_{kh}) = P_{0,h}\gamma_{1,\epsilon}(X^{\epsilon}_0) - P_{(n-2)h,(n-1)h}\gamma_{n-1,\epsilon}(X^{\epsilon}_{(n-2)h}) + \sum_{k=1}^{n-2} \gamma_{k,\epsilon}(X^{\epsilon}_{kh}) - P_{(k-1)h,kh}\gamma_{k,\epsilon}(X^{\epsilon}_{(k-1)h})$$

Now, since $f_{(n-1)h}(X^{\epsilon}_{(n-1)h}) = \gamma_{n-1,\epsilon}(X^{\epsilon}_{(n-1)h})$ and

$$f_0(X^{\epsilon}_0) = \gamma_{0,\epsilon}(X^{\epsilon}_0) - P_{0,h}\gamma_{1,\epsilon}(X^{\epsilon}_0),$$

we conclude. The third statement follows from $\mathbb{E}\Big[\gamma_{k,\epsilon}(X^{\epsilon}_{kh}) - P_{(k-1)h,kh}\gamma_{k,\epsilon}(X^{\epsilon}_{(k-1)h}) \mid \mathcal{F}_{(k-1)h}\Big] = 0$ for $k \in \{1, \ldots, n-1\}$and the first statement combined with Lemma 14 (for the lemma's p sufficiently large) and the fact that $\sup_{t\in[0,1]} \|x^{\star}_t\| < \infty$ from Lemma 68, which establishes that for any $i \in \{0, \ldots, n-1\}$, $\mathbb{E}(|M_{i,\epsilon}|) < \infty$. □

In what follows we let $M_\epsilon := M_{n-1,\epsilon}$ where the latter is defined in Lemma 37. The following proposition will be used to establish that one can obtain the desired quantitative CLT bounds by focusing on the martingale approximation and the appropriate control of vanishing terms.

Proposition 38. *For any $\varepsilon_1, \varepsilon_2 > 0$ and $\epsilon > 0$ such that $\upsilon(\epsilon) > 0$,*

$$\begin{aligned}\sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[S_{\epsilon,h}/\sqrt{\epsilon\upsilon(\epsilon)} \leqslant w\big] - \Phi(w)\big| \leqslant \sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[M_\epsilon \leqslant w\big] - \Phi(w)\big| + \mathbb{P}\big[|B_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)} > \varepsilon_1/2\big] \\ + \mathbb{P}\big[h|\gamma_{0,\epsilon}(X^{\epsilon}_0)|/\sqrt{\epsilon\upsilon(\epsilon)} > \varepsilon_1/2\big] + (2\pi)^{-1/2}\varepsilon_1,\end{aligned}$$

and

$$\begin{aligned}\sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[S_{\epsilon,h}/\sqrt{\epsilon\sigma^2_\ell} \leqslant w\big] - \Phi(w)\big| \leqslant 2\sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[S_{\epsilon,h}/\sqrt{\epsilon\upsilon(\epsilon)} \leqslant w\big] - \Phi(w)\big| + 1 - \Phi(\varepsilon_1\varepsilon_2^{-1}) \\ + \mathbb{P}\big[\big|\upsilon^{1/2}(\epsilon)/\sigma - 1\big| > \varepsilon_2\big] + (2\pi)^{-1/2}\varepsilon_1.\end{aligned}$$

Proof. We have the general result that for $\varepsilon > 0$ and two random variables Z_1, Z_2

$$\mathbb{P}\big[Z_1 \leqslant w - \varepsilon\big] - \mathbb{P}\big[|Z_2| > \varepsilon\big] \leqslant \mathbb{P}\big[Z_1 + Z_2 \leqslant w\big] \leqslant \mathbb{P}\big[Z_1 \leqslant w + \varepsilon\big] + \mathbb{P}\big[|Z_2| > \varepsilon\big],$$

and therefore

$$\begin{aligned}\mathbb{P}\big[Z_1 \leqslant w - \varepsilon\big] - \Phi(w-\varepsilon) + \Phi(w-\varepsilon) - \Phi(w) - \mathbb{P}\big[|Z_2| > \varepsilon\big] \leqslant \mathbb{P}\big[Z_1 + Z_2 \leqslant w\big] - \Phi(w) \\ \leqslant \mathbb{P}\big[Z_1 \leqslant w + \varepsilon\big] - \Phi(w+\varepsilon) + \Phi(w+\varepsilon) - \Phi(w) + \mathbb{P}\big[|Z_2| > \varepsilon\big].\end{aligned}$$

Now notice that $\max_{a\in\{\varepsilon,-\varepsilon\}}\big|\Phi(w+a)-\Phi(w)\big|\leqslant(2\pi)^{-1/2}\varepsilon$ and conclude that

$$\sup_{w\in\mathbb{R}}\big|\mathbb{P}\big[Z_1+Z_2\leqslant w\big]-\Phi(w)\big|\leqslant\sup_{w'\in\mathbb{R}}\big|\mathbb{P}\big[Z_1\leqslant w'\big]-\Phi(w')\big|+\mathbb{P}\big[|Z_2|>\varepsilon\big]+(2\pi)^{-1/2}\varepsilon.$$

We have

$$S_{\epsilon,h}/\sqrt{\epsilon\upsilon(\epsilon)}=(h\gamma_{0,\epsilon}(X_0^\epsilon)-B_{\epsilon,h})/\sqrt{\epsilon\upsilon(\epsilon)}+M_\epsilon,$$

and

$$S_{\epsilon,h}/\sqrt{\epsilon\sigma_\ell^2}=S_{\epsilon,h}/\sqrt{\epsilon\upsilon(\epsilon)}+\epsilon^{-1/2}S_{\epsilon,h}\big(\sigma_\ell^{-1}-\upsilon^{-1/2}(\epsilon)\big).$$

We can apply the above general inequality to these two identities in turn. In the first case we also note the fact that $\mathbb{P}\big[|Z_1+Z_2|>\varepsilon\big]\leqslant\mathbb{P}\big[|Z_1|+|Z_2|>\varepsilon\big]\leqslant\mathbb{P}\big[|Z_1|>\varepsilon/2\big]+\mathbb{P}\big[|Z_2|>\varepsilon/2\big]$. In the second case we have that, in general, for non-negative random variables Z_1,Z_2 and any $\varepsilon_1,\varepsilon_2>0$

$$\mathbb{P}\big[Z_1Z_2>\varepsilon_1\big]\leqslant\mathbb{P}\big[Z_1>\varepsilon_1\varepsilon_2^{-1}\big]+\mathbb{P}\big[Z_2>\varepsilon_2\big]$$

and therefore

$$\mathbb{P}\big[\epsilon^{-1/2}|S_{\epsilon,h}||\sigma_\ell^{-1}-\upsilon^{-1/2}(\epsilon)|>\varepsilon_1\big]\leqslant\mathbb{P}\big[|S_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)}>\varepsilon_1\varepsilon_2^{-1}\big]+\mathbb{P}\big[\big|\upsilon^{1/2}(\epsilon)/\sigma_\ell-1\big|>\varepsilon_2\big].$$

Finally

$$\mathbb{P}\big[|S_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)}>\varepsilon_1\varepsilon_2^{-1}\big]=1-\mathbb{P}\big[|S_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)}\leqslant\varepsilon_1\varepsilon_2^{-1}\big]+\Phi(\varepsilon_1\varepsilon_2^{-1})-\Phi(\varepsilon_1\varepsilon_2^{-1}),$$

from which we conclude. □

The following lemmata establish quantitative bounds for some of the vanishing terms appearing in one of the upper bounds in Proposition 38. A quantitative bound for $\mathbb{P}\big[\big|\upsilon^{1/2}(\epsilon)/\sigma_\ell-1\big|>\varepsilon_2\big]$ is established latter in Corollary 48. Altogether these results justify the focus on the martingale term in Section 5.3.

Lemma 39. *Let $p\geqslant1$ and $f\in C^p_{0,2}([0,1]\times\mathbb{R}^d)$, and assume (A1-5). Then*

1. *for any $\varepsilon_1>0$, $\ell\geqslant0$, $\daleth>1$ and $\epsilon,h>0$ such that $\daleth^{-1}\leqslant1-Kh\epsilon^{-1}/2$,*

$$\mathbb{P}\big[|B_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)}>\varepsilon_1/2\big]\leqslant\mathbb{I}\{F>\upsilon(\epsilon)^{1/2}\epsilon^{-1/2}\varepsilon_1\},$$

   where

$$F:=2\|\nabla f\|_p\cdot\sup_{s\in[0,1]}\pi_s\bar V^{(2[p\vee p_0]+1/2)}\left\{K^{-2}\|\nabla\phi\|_{p_0}+\alpha_p\mu_0\bar V^{(p+1/2)}\frac{\daleth}{K}\right\},$$

2. *further assuming (A10-11), we deduce that for any $c\in(0,1/2)$ and the choice $\varepsilon_1(d)=C\epsilon(d)^c$ there exists $a_0>0$ and $d_0\in\mathbb{N}$ such that with $\epsilon(d)=Cd^{-a}$, for $a\geqslant a_0$ and $d\geqslant d_0$*

$$\mathbb{P}\big[|B_{\epsilon(d),h(d)}|/\sqrt{\epsilon(d)\upsilon_d\big(\epsilon(d)\big)}>\varepsilon_1(d)/2\big]=0.$$

Proof. From Lemma 58,

$$\begin{aligned}|B_{\epsilon,h}|&\leqslant C\|\nabla f\|_p\cdot\sup_{s\in[0,1]}\pi_s\bar V^{(2[p\vee p_0]+1/2)}\Big\{K^{-2}\|\nabla\phi\|_{p_0}+\alpha_p\mu_0\bar V^{(p+1/2)}\frac{h\epsilon^{-1}}{1-\exp\big(-Kh\epsilon^{-1}\big)}\Big\}\epsilon,\\&\leqslant F/2\epsilon.\end{aligned}$$

where we have used Lemma 60. Further

$$\begin{aligned}\mathbb{P}\big[|B_{\epsilon,h}|/\sqrt{\epsilon\upsilon(\epsilon)}>\varepsilon_1/2\big]&=\mathbb{I}\{2|B_{\epsilon,h}|>\sqrt{\epsilon\upsilon(\epsilon)}\varepsilon_1\},\\&\leqslant\mathbb{I}\{F>\sqrt{\upsilon(\epsilon)}\epsilon^{-1/2}\varepsilon_1\}.\end{aligned}$$

For the second part we first notice that from (A10) $F(d)$ grows at most polynomially in d, say $F(d)\leqslant Cd^f$. Then

$$\upsilon_d\big(\epsilon(d)\big)^{1/2}\epsilon(d)^{-1/2}\varepsilon_1(d)=\big[\sigma_\ell^2(d)+\upsilon_d\big(\epsilon(d)\big)-\sigma_\ell^2(d)\big]^{1/2}\epsilon(d)^{-1/2}\varepsilon_1(d).$$

From (A10) $\sigma_\ell^2(d) \geqslant Cd^{-r}$ for some $r > 0$ and from Theorem 47 there exists $a_0 > 0$ such that for any $a > 0$ one can make $\upsilon_d\big(\epsilon(d)\big) - \sigma_\ell^2(d)$ vanish faster than d^{-r}. Let $a_1 \geqslant a_0$, then for d sufficiently large,

$$\upsilon_d\big(\epsilon(d)\big)^{1/2}\epsilon(d)^{-1/2}\varepsilon_1(d) \geqslant \big[\sigma_\ell^2(d)/2\big]^{1/2}\epsilon(d)^{-1/2}\varepsilon_1(d)$$

Now choose $\varepsilon_1(d) = \epsilon(d)^c$ with $c < 1/2$ and $a > a_1 \vee \big[(r+f)/(1/2-c)\big]$ and $\epsilon(d) = Cd^{-a}$, then $\sigma_\ell(d)\epsilon(d)^{-1/2+c}F(d)^{-1}$ diverges and we conclude. ☐

Lemma 40. *Assume (A1-5) and (A10). Then*

1. *there exists $C > 0$ such that for any $\epsilon, \varepsilon_1, h > 0$ such that $\upsilon(\epsilon) > 0$ and for some $\beth > 1$ and $\beth^{-1} \leqslant 1 - Kh\epsilon^{-1}/2$*

$$\mathbb{P}\big[h|\gamma_{0,\epsilon}(X_0^\epsilon)|/\sqrt{\epsilon\upsilon(\epsilon)} > \varepsilon_1/2\big] \leqslant C\left(\frac{\alpha_p}{\epsilon^{-1/2}\varepsilon_1\sqrt{\upsilon(\epsilon)}}\frac{\beth\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\mu_0\bar{V}^{(p+1/2)}\right),$$

2. *for any $c \in (0,1/2)$ and the choice $\varepsilon_1(d) = C\epsilon(d)^c$ there exists $a_0 > 0$ sufficiently large such that for any $a > a_0$ and $\epsilon(d) = Cd^{-a}$*

$$\lim_{d\to\infty}\mathbb{P}\big[h(d)|\gamma_{0,\epsilon(d)}(X_0^{\epsilon(d)})|/\sqrt{\epsilon(d)\upsilon_d\big(\epsilon(d)\big)} > \varepsilon_1(d)/2\big] = 0$$

Proof. From Markov's inequality, Lemma 37 and Lemma 57

$$\mathbb{P}\big[h|\gamma_{0,\epsilon}(X_0^\epsilon)|/\sqrt{\epsilon\upsilon(\epsilon)} > \varepsilon_1/2\big] \leqslant 2\frac{h}{\varepsilon_1\sqrt{\epsilon\upsilon(\epsilon)}}\mu_0\big(|\gamma_{0,\epsilon}|\big)$$
$$\leqslant C\frac{\alpha_p\epsilon^{1/2}}{\varepsilon_1\sqrt{\upsilon(\epsilon)}}\frac{\|\nabla f\|_p h\epsilon^{-1}}{1-\exp\big(-K\epsilon^{-1}h\big)}\left(\mu_0\bar{V}^{(p+1/2)}\right)^2$$
$$\leqslant C\frac{\alpha_p\epsilon^{1/2}}{\varepsilon_1\sqrt{\upsilon(\epsilon)}}\frac{\beth\|\nabla f\|_p}{K}\left(\mu_0\bar{V}^{(p+1/2)}\right)^2.$$

The proof is now similar to that of the second part of Lemma 39. ☐

5.3 Quantitative bound in the CLT for the Martingale approximation

We now state an intermediate result which motivates subsequent developments to prove the quantitative bounds in Theorem 36.

Theorem 41. *Let $p \geqslant 1$ and for any $d \in \mathbb{N}$, let $(X_t^\epsilon(d))_{t\in[0,1]}$ be as defined in (90) and $f^{(d)} \in C_{1,2}^p([0,1]\times\mathbb{R}^d)$. Assume that for any $d \in \mathbb{N}$ (A1-5) and (A10) hold. Let $M_\epsilon := M_{n-1,\epsilon}$ where the latter is defined in Lemma 37. Then for any $\ell \geqslant 0$ there exists $a > 0$ such that with $\epsilon(d) = O(d^{-a})$ and $d \mapsto h(d)$ satisfying (A11)*

$$\lim_{d\to\infty}\sup_{w\in\mathbb{R}}\big|\mathbb{P}\big[M_{\epsilon(d)} \leqslant w\big] - \Phi(w)\big| = 0.$$

Proof. The proof relies on the upper bound established in Proposition 42 and bounds for A_ϵ, B_ϵ and C_ϵ which can be deduced from Lemma 43 and 45, and Theorem 47. More precisely, choose $\kappa > c - 1$, where c is given in (A11). For A_ϵ: from (A10) and Lemma (70) one deduces that the bound on $\mathbb{E}\big[\big|D_\epsilon\big|^{1+\kappa}\big]^{1/(1+\kappa)}$ in Lemma 45 grows at most as a polynomial of d. (A10) implies the existence of $r > 0$ such that $\sigma_\ell^2(d) \geqslant Cd^{-r}$ and Theorem 47 implies the existence of $a_0, d_0 > 0$ such that for any $a \geqslant a_0$ and $d \geqslant d_0$

$$\sigma_\ell^2(d) + \upsilon\big(\epsilon(d)\big) - \sigma_\ell^2(d) \geqslant \sigma_\ell^2(d)/2. \tag{93}$$

Now, again from Theorem 47 we can choose b sufficiently large (and hence a sufficiently large) such that

$$\big|\upsilon_d\big(\epsilon(d)\big) - \sigma_\ell^2(d)\big|\mathbb{E}\big[\big|D_\epsilon\big|^{1+\kappa}\big]^{1/(1+\kappa)}\sigma_\ell^{-4}(d)$$

vanishes. Therefore $\lim_{d\to 0} A_{\epsilon(d)} = 0$. For B_ϵ we use Lemma 43, its Corollary, the lower bound ((93)) and Corollary 3 of Theorem 1 to conclude that for $a \geqslant a_0$ sufficiently large $\lim_{d\to 0} B_{\epsilon(d)} = 0$. Finally $\lim_{d\to 0} C_{\epsilon(d)} = 0$ follows from Lemma 45 and its Corollary 46, since we have assumed $\kappa > c-1$ in order to cover the scenario $\ell = 0$. □

Let

$$D_\epsilon := \epsilon^{-1} h^2 \sum_{k=0}^{n-1} \mathbb{E}\big[\xi_{k,\epsilon}^2 | \mathcal{F}_{(k-1)h}\big],$$

where $\xi_{k,\epsilon}$is as in Lemma (37).

Proposition 42. *For any $\kappa > 0$ that there exists a finite $\mathscr{C}_\kappa > 0$, dependent on κ only, such that*

$$\sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[M_\epsilon \leqslant w\big] - \Phi(w)\big| \leqslant \mathscr{C}_\kappa \Big\{ \Big(A_\epsilon + B_\epsilon\Big)^{1+\kappa} + C_\epsilon \Big\}^{1/(3+2\kappa)},$$

where

$$\begin{aligned}
A_\epsilon &:= \big|\upsilon(\epsilon) - \sigma_\ell^2\big| \Big[1 + \mathbb{E}\big[|D_\epsilon|^{1+\kappa}\big]^{1/(1+\kappa)} / \upsilon(\epsilon)\Big] / \sigma_\ell^2, \\
B_\epsilon &:= \mathbb{E}\Big[\big|D_\epsilon - \upsilon(\epsilon)\big|^{1+\kappa}\Big]^{1/(1+\kappa)} / \sigma_\ell^2, \\
C_\epsilon &:= (\epsilon^{-1} h^2 / \upsilon(\epsilon))^{(1+\kappa)} \sum_{i=0}^{n-1} \mathbb{E}\Big[\big|\xi_{i,\epsilon}\big|^{2(1+\kappa)}\Big].
\end{aligned}$$

Proof. $\Delta_\epsilon := \sup_{w\in\mathbb{R}} \big|\mathbb{P}\big[M_\epsilon \leqslant w\big] - \Phi(w)\big|$. From [18, Theorem 1] we have

$$\Delta_\epsilon \leqslant \mathscr{C}_\kappa \Big\{ \mathbb{E}\Big[\big|D_\epsilon/\upsilon(\epsilon) - 1\big|^{1+\kappa}\Big] + (\epsilon^{-1} h^2/\upsilon(\epsilon))^{(1+\kappa)} \sum_{i=0}^{n-1} \mathbb{E}\Big[\big|\xi_{i,\epsilon}\big|^{2(1+\kappa)}\Big] \Big\}^{1/(3+2\kappa)}.$$

We upper bound the first term between braces using Minkowski's inequality

$$\begin{aligned}
\mathbb{E}\Big[\big|D_\epsilon/\upsilon(\epsilon) - 1\big|^{1+\kappa}\Big]^{1/(1+\kappa)} &\leqslant \mathbb{E}\Big[\big|D_\epsilon/\sigma_\ell^2 - 1\big|^{1+\kappa}\Big]^{1/(1+\kappa)} + \mathbb{E}\Big[\big|D_\epsilon \left(\upsilon^{-1}(\epsilon) - \sigma_\ell^{-2}\right)\big|^{1+\kappa}\Big]^{1/(1+\kappa)} \\
&\leqslant \mathbb{E}\Big[\big|D_\epsilon - \sigma_\ell^2\big|^{1+\kappa}\Big]^{1/(1+\kappa)} / \sigma_\ell^2 + \big|\upsilon^{-1}(\epsilon) - \sigma_\ell^{-2}\big| \mathbb{E}\Big[\big|D_\epsilon\big|^{1+\kappa}\Big]^{1/(1+\kappa)},
\end{aligned}$$

and further

$$\mathbb{E}\Big[\big|D_\epsilon - \sigma_\ell^2\big|^{1+\kappa}\Big]^{1/(1+\kappa)} \leqslant \mathbb{E}\Big[\big|D_\epsilon - \upsilon(\epsilon)\big|^{1+\kappa}\Big]^{1/(1+\kappa)} + \big|\upsilon(\epsilon) - \sigma_\ell^2\big|,$$

from which we conclude. □

We need to find explicit upper bounds for the three terms above. In the next two propositions we will make use of the following alternative expression for D_ϵ

$$\begin{aligned}
D_\epsilon &= \epsilon^{-1} h^2 \sum_{k=1}^{n-1} \mathbb{E}\Big[\gamma_{k,\epsilon}^2(X_{kh}^\epsilon) - \big(P_{(k-1)h,kh}^\epsilon \gamma_{k,\epsilon}(X_{(k-1)h}^\epsilon)\big)^2 \mid \mathcal{F}_{(k-1)h}\Big] \\
&= \epsilon^{-1} h^2 \Big\{ P_{(n-2)h,(n-1)h}^\epsilon \gamma_{n-1,\epsilon}^2(X_{(n-2)h}^\epsilon) - \big[P_{0,h}^\epsilon \gamma_{1,\epsilon}(X_0^\epsilon)\big]^2 \\
&\qquad + \sum_{k=1}^{n-2} \mathbb{E}\Big[\gamma_{k,\epsilon}^2(X_{kh}^\epsilon) - \big(P_{kh,(k+1)h}^\epsilon \gamma_{k+1,\epsilon}(X_{kh}^\epsilon)\big)^2 \mid \mathcal{F}_{(k-1)h}\Big] \Big\} \\
&= \epsilon^{-1} h^2 \Big\{ P_{(n-2)h,(n-1)h}^\epsilon f_{n-1}^2(X_{(n-2)h}^\epsilon) - \big[P_{0,h}^\epsilon \gamma_{1,\epsilon}(X_0^\epsilon)\big]^2 \\
&\qquad + \sum_{k=1}^{n-2} \mathbb{E}\Big[f_{kh}(X_{kh}^\epsilon)\big(\gamma_{k,\epsilon}(X_{kh}^\epsilon) + P_{kh,(k+1)h}^\epsilon \gamma_{k+1,\epsilon}(X_{kh}^\epsilon)\big) \mid \mathcal{F}_{(k-1)h}\Big] \Big\}.
\end{aligned}$$

and we let

$$\tilde{D}_\epsilon := \epsilon^{-1}h^2 \sum_{k=1}^{n-2} \mathbb{E}\big[f_{kh}(X^\epsilon_{kh})\big(\gamma_{k,\epsilon}(X^\epsilon_{kh}) + P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(X^\epsilon_{kh})\big) \mid \mathcal{F}_{(k-1)h}\big]$$

Lemma 43. *For any $\kappa > 1$, $r > (1+\kappa)/2$ and with $m := \big((1+\kappa)r - 2\big)/(r-1)$ we have*

$$\begin{aligned}\|D_\epsilon - \upsilon(\epsilon)\|_{L_{1+\kappa}} \leqslant C\big(\|\tilde{D}_\epsilon - \mathbb{E}\big(\tilde{D}_\epsilon\big)\|_{L_2}\big)^{1/[(1+\kappa)r]} &\left(\alpha_{2pm}^{1/m}\big(\mu_0\bar{V}^{(2pm)}\big)^{1/m} + \alpha_{2p}\mu_0\bar{V}^{(2p)}\right)^{m/(1+\kappa)} \\ &\times \left(\alpha_p\alpha_{2p+1/2}\frac{\daleth\|f\|_p\|\nabla f\|_p}{K}\pi_0\bar{V}^{(p+1/2)}\right)^{m/(1+\kappa)} \\ &+ C\epsilon\left(\alpha_p\frac{\daleth\|\nabla f\|_p}{K}\pi_0\bar{V}^{(p+1/2)}\right)^2 \cdot \left(\alpha_{(1+\kappa)(2p+1)}\mu_0\bar{V}^{([1+\kappa][2p+1])}\right)^{1/(1+\kappa)} \\ &+ C\epsilon^{-1}h^2\alpha_{2p}\|f\|_p^2\left(\alpha_{2p(1+\kappa)}\mu_0\bar{V}^{(2p[1+\kappa])}\right)^{1/(1+\kappa)}.\end{aligned}$$

Proof. From Lemma 37 we know that for $k \in \{0,\ldots,n-1\}$ $\big(\gamma_{k,\epsilon}, P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}\big) \in C_2^p(\mathbb{R}^d)$, and as a result, using Lemma 56, $\left(\gamma^2_{k,\epsilon}, \big(P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}\big)^2\right) \in C_2^{2p}(\mathbb{R}^d)$ and from Proposition 15 we have that $P^\epsilon_{(k-1)h,kh}\left(\gamma^2_{k,\epsilon}\right), P^\epsilon_{(k-1)h,kh}\left(\big(P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}\big)^2\right) \in C_2^{2p}(\mathbb{R}^d)$. Further, from Lemma 37, we have for $\daleth > 1$ and $\daleth^{-1} < 1 - Kh\epsilon^{-1}/2$

$$\big|P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(x)\big| \vee \big|\gamma_{k,\epsilon}(x)\big| \leqslant C\epsilon h^{-1}\alpha_p\frac{\daleth\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)} \cdot \bar{V}^{(p+1/2)}(x) \tag{94}$$

and therefore from Lemma 56 and Lemma 14

$$\begin{aligned}P^\epsilon_{(k-1)h,kh}\Big(\big|f_{kh}\gamma_{k,\epsilon}\big|\Big)(x) \vee P^\epsilon_{(k-1)h,kh}\Big(\big|f_{kh}P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}\big|\Big)(x)& \\ \leqslant C\epsilon h^{-1}\alpha_p\frac{\daleth\|f\|_p\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\alpha_{2p+1/2}\bar{V}^{(2p+1/2)}(x).&\end{aligned}$$

We deduce that for $q > 1$

$$\begin{aligned}&\epsilon^{-1}h^2\|\big[P^\epsilon_{0,h}\gamma_{1,\epsilon}(X^\epsilon_0)\big]^2 - \mathbb{E}\left(\big[P^\epsilon_{0,h}\gamma_{1,\epsilon}(X^\epsilon_0)\big]^2\right)\|_{L_q} \\ &\leqslant C\epsilon\left(\alpha_p\frac{\daleth\|f\|_p\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\right)^2 \cdot \left(\alpha_{q(2p+1)}\mu_0\bar{V}^{(q[2p+1])}\right)^{1/q}\end{aligned}$$

Further

$$P^\epsilon_{(n-2)h,(n-1)h}f^2_{n-1}(x) \leqslant \|f\|_p^2\alpha_{2p}\bar{V}^{(2p)}(x) \tag{95}$$

and therefore, for $q > 1$

$$\begin{aligned}\epsilon^{-1}h^2\|P^\epsilon_{(n-2)h,(n-1)h}f^2_{n-1}(X^\epsilon_{(n-2)h}) - \mathbb{E}\left(P^\epsilon_{(n-2)h,(n-1)h}f^2_{n-1}(X^\epsilon_{(n-2)h})\right)\|_{L_q}& \\ \leqslant C\epsilon^{-1}h^2\alpha_{2p}\|f\|_p^2\left(\alpha_{2pq}\mu_0\bar{V}^{(2pq)}\right)^{1/q}.&\end{aligned}$$

Now

$$\begin{aligned}\|D_\epsilon - \upsilon(\epsilon)\|_{L_{1+\kappa}} \leqslant \|\tilde{D}_\epsilon - \mathbb{E}\big(\tilde{D}_\epsilon\big)\|_{L_{1+\kappa}} &+ \epsilon^{-1}h^2\|\big[P^\epsilon_{0,h}\gamma_{1,\epsilon}(X^\epsilon_0)\big]^2 - \mathbb{E}\left(\big[P^\epsilon_{0,h}\gamma_{1,\epsilon}(X^\epsilon_0)\big]^2\right)\|_{L_{1+\kappa}} \\ &+ \epsilon^{-1}h^2\|P^\epsilon_{(n-2)h,(n-1)h}f^2_{n-1}(X^\epsilon_{(n-2)h}) - \mathbb{E}\left(P^\epsilon_{(n-2)h,(n-1)h}f^2_{n-1}(X^\epsilon_{(n-2)h})\right)\|_{L_{1+\kappa}}.\end{aligned}$$

Now we apply Lemma 55 for the sum of terms $h\epsilon^{-1}\mathbb{E}\big[f_{kh}(X^\epsilon_{kh})\big(\gamma_{k,\epsilon}(X^\epsilon_{kh}) + P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(X^\epsilon_{kh})\big) \mid \mathcal{F}_{(k-1)h}\big]$, $q = 1+\kappa$, $r, m > 0$ such that $r > q/2 > 1$ and $m = (qr-2)/(r-1)$

$$\begin{aligned}\|\tilde{D}_\epsilon - \mathbb{E}\big(\tilde{D}_\epsilon\big)\|_{L_q} \leqslant C\big(\|\tilde{D}_\epsilon - \mathbb{E}\big(\tilde{D}_\epsilon\big)\|_{L_2}\big)^{2/(qr)} &\left(\alpha_{2pm}^{1/m}\big(\mu_0\bar{V}^{(2pm)}\big)^{1/m} + \alpha_{2p}\mu_0\bar{V}^{(2p)}\right)^{1-2/(qr)} \\ &\times \left(\alpha_p\alpha_{2p+1/2}\frac{\daleth\|f\|_p\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\right)^{1-2/(qr)}.\end{aligned}$$

We can conclude. ∎

Corollary 44. *From Theorem 1 we can conclude that under (A10) and (A11), for any $\kappa > 1$, there exist $r_1, r_2 > 0$ such that $\|D_{\epsilon(d)} - \upsilon(\epsilon(d))\|_{L_{1+\kappa}} \leqslant C d^{r_1} \epsilon^{r_2}(d)$.*

Lemma 45. *For any $\kappa > 0$ there exist C dependent on κ only, such that for any $\beth > 1$ and $K, \epsilon, h > 0$ such that $\beth^{-1} \leqslant 1 - Kh\epsilon^{-1}/2$ and $\ell \geqslant 0$, then*

$$C_\epsilon \leqslant C\upsilon(\epsilon)^{-(1+\kappa)}(\epsilon h^{-1+\kappa/(1+\kappa)})^{1+\kappa}\Big\{\alpha_p \frac{\beth\|\nabla f\|_p}{K}\mu_0 \bar{V}^{(p+1/2)}\Big\}^{2(1+\kappa)} \cdot \alpha_{2(1+\kappa)(p+1/2)}\mu_0 \bar{V}^{(2[1+\kappa][p+1/2])},$$

and

$$\begin{aligned}\mathbb{E}\Big[\big|D_\epsilon\big|^{1+\kappa}\Big]^{1/(1+\kappa)} &\leqslant C\alpha_p\alpha_{2p+1/2}\alpha_{(1+\kappa)(2p+1/2)}\frac{\beth\|\nabla f\|_p\|f\|_p}{K}\mu_0\bar{V}^{(p+1/2)} \cdot \big\{\mu_0\bar{V}^{([1+\kappa][2p+1/2])}\big\}^{1/(1+\kappa)}.\\ &+ C\epsilon\left(\alpha_p\frac{\beth\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\right)^2 \cdot \left(\alpha_{(1+\kappa)(2p+1)}\mu_0\bar{V}^{([1+\kappa][2p+1])}\right)^{1/(1+\kappa)}\\ &+ C\epsilon^{-1}h^2\alpha_{2p}\|f\|_p^2\left(\alpha_{2p(1+\kappa)}\mu_0\bar{V}^{(2p[1+\kappa])}\right)^{1/(1+\kappa)}.\end{aligned}$$

Corollary 46. *With $h(\epsilon) = C\epsilon^\iota$ where $\iota \geqslant 1$*

$$C_\epsilon \leqslant C\upsilon(\epsilon)^{-(1+\kappa)}\epsilon^{1+\kappa-\iota}\Big\{\alpha_p\frac{\beth\|\nabla f\|_p}{K}\mu_0\bar{V}^{(p+1/2)}\Big\}^{2(1+\kappa)} \cdot \alpha_{2(1+\kappa)(p+1/2)}\mu_0\bar{V}^{(2[1+\kappa][p+1/2])}.$$

Proof. For C_ϵ we first apply Minkowski's inequality followed with Lemma 37, Jensen's inequality and Lemma 14

$$\begin{aligned}\mathbb{E}\Big[\xi_{k,\epsilon}^{2(1+\kappa)}\Big]^{1/(2+2\kappa)} &\leqslant \mathbb{E}\Big[\big|\gamma_{k,\epsilon}(X_{kh}^\epsilon)\big|^{2(1+\kappa)}\Big]^{1/(2+2\kappa)} + \mathbb{E}\Big[\big|P^\epsilon_{(k-1)h,kh}\gamma_{k,\epsilon}(X^\epsilon_{(k-1)h})\big|^{2(1+\kappa)}\Big]^{1/(2+2\kappa)}\\ &\leqslant C\alpha_p\frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)} \cdot \mathbb{E}\Big[\bar{V}^{(p+1/2)}(X^\epsilon_{(k-1)h})^{2(1+\kappa)}\Big]^{1/(2+2\kappa)}\\ &\leqslant C\alpha_p\frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)} \cdot \Big[\alpha_{2(1+\kappa)(p+1/2)}\mu_0\bar{V}^{(2[1+\kappa][p+1/2])}\Big]^{1/(2+2\kappa)}.\end{aligned}$$

Therefore

$$C_\epsilon \leqslant C\upsilon(\epsilon)^{-(1+\kappa)}h^\kappa\Big\{\alpha_p\frac{\|\nabla f\|_p(\epsilon^{-1}h)^{1/2}}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)}\Big\}^{2(1+\kappa)} \cdot \alpha_{2(1+\kappa)(p+1/2)}\mu_0\bar{V}^{(2[1+\kappa][p+1/2])}$$

Now from Lemma (60), for $1/\beth \leqslant 1 - Kh\epsilon^{-1}/2$

$$\frac{(\epsilon^{-1}h)^{1/2}}{1-\exp\big(-K\epsilon^{-1}h\big)} \leqslant \frac{\beth}{K}(\epsilon h^{-1})^{1/2}$$

and the term dependent on ϵ and h in the upper bound is indeed of the form $h^\kappa(\epsilon h^{-1})^{1+\kappa} = (\epsilon h^{-1+\kappa/(1+\kappa)})^{1+\kappa}$. For the second statement, from Lemma 37

$$\begin{aligned}&\Big|\mathbb{E}\big[f_{kh}(X^\epsilon_{kh})\big(\gamma_{k,\epsilon}(X^\epsilon_{kh}) + P^\epsilon_{kh,(k+1)h}\gamma_{k+1,\epsilon}(X^\epsilon_{kh})\big) \mid \mathcal{F}_{(k-1)h}\big]\Big|\\ &\qquad\leqslant C\alpha_p\frac{\|\nabla f\|_p\|f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)} \cdot P_{(k-1)h,kh}\big(\bar{V}^{(p+1/2)}\bar{V}^{(p)}\big)(X^\epsilon_{(k-1)h})\\ &\qquad\qquad\leqslant C\alpha_p\alpha_{2p+1/2}\frac{\|\nabla f\|_p\|f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)} \cdot \bar{V}^{(2p+1/2)}(X^\epsilon_{(k-1)h}),\end{aligned}$$

where we have used Lemmas 56 and 14. Consequently

$$\begin{aligned}\mathbb{E}\big[\big|\tilde{D}_\epsilon\big|^{1+\kappa}\big]^{1/(1+\kappa)} &\leqslant C\alpha_p\alpha_{2p+1/2}\frac{\|\nabla f\|_p\|f\|_p\epsilon^{-1}h}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)}h^{-1}\sum_{k=1}^{n-2}\mathbb{E}\big[\big|\bar{V}^{(2p+1/2)}(X^\epsilon_{(k-1)h})\big|^{1+\kappa}\big]^{1/(1+\kappa)}\\ &\leqslant C\alpha_p\alpha_{2p+1/2}\alpha_{(1+\kappa)(2p+1/2)}\frac{\|\nabla f\|_p\|f\|_p\epsilon^{-1}h}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)} \cdot \big\{\mu_0\bar{V}^{([1+\kappa][2p+1/2])}\big\}^{1/(1+\kappa)}\end{aligned}$$

and from (94) and (95) in the proof of Lemma (43) we can conclude. $\square$

5.4 Quantitative bound on the convergence of the CLT constants

For $\epsilon > 0$, and $x \in \mathbb{R}^d$ we define for $k \in \{0, \ldots, n-1\}$

$$\eta_{k,\epsilon}(x) := \mathbb{E}\left[\sum_{i=0}^{n-1} \bar{f}_{kh}(Y_{ih}^{s,\epsilon}) \mid Y_0^s = x\right] = \sum_{i=0}^{n-1} Q_{ih\epsilon^{-1}}^{kh} \bar{f}_{kh}(x)$$

and for $s \in [0,1]$

$$g_s(x) := \begin{cases} \ell \sum_{k=0}^{\infty} Q_{k\ell}^s \bar{f}_s(x) & \text{if } \ell = \epsilon^{-1}h > 0 \\ \int_0^\infty Q_t^s \bar{f}_s(x)\mathrm{d}t & \text{if } \ell = 0 \end{cases}.$$

Note that it is not difficult to show that with our assumptions, for $\ell \geqslant 0$ and $s \in [0,1]$,

$$\varsigma_\ell(s) = 2\mathbb{E}\Big[\bar{f}_s(Y_0^s) g_s(Y_0^s)\Big] - \ell \mathrm{var}\big(f_s(Y_0^s)\big). \tag{96}$$

Before presenting our results, we discuss a presentational point. The term $1/\big[1-\exp(-Kh\epsilon^{-1})\big]$ appears repeatedly in a number of upper bounds. This term will not pose any problem whenever $K(d)h(d)\epsilon^{-1}(d) \geqslant z$, for say $d \geqslant d_0$ and some $z > 0$. Our statements therefore focus on the more "difficult" scenario where $\limsup_{d\to\infty} K(d)h(d)\epsilon^{-1}(d) = 0$, but one should bear in mind that similar conclusions can be drawn in the former "easier" scenario.

Theorem 47. *Assume (A1-5) and (A10). Then, with the following choices*

1. *for any $\ell > 0$ such that for some $\gimel > 1$, $\gimel^{-1} \leqslant 1 - K(d)\ell/2$ for $d \geqslant d_0$ for some $d_0 \in \mathbb{N}$ we set $h(d) := \ell\epsilon(d)$,*
2. *for $\ell = 0$ we set $h(d) = C\epsilon^c(d)$ for some $c > 1$,*

for any $b > 0$ there exists $a_0 > 0$ such that for any $a \geqslant a_0$ and $\epsilon(d) = Cd^{-a}$ we have

$$\limsup_{d\to\infty} d^b \big|\upsilon_d\big(\epsilon(d)\big) - \sigma_\ell^2(d)\big| < \infty.$$

Corollary 48. *With Lemma 38 in mind, we have*

$$\begin{aligned} \mathbb{P}\big[\big|\upsilon^{1/2}\big(\epsilon(d)\big)/\sigma_\ell(d) - 1\big| > \varepsilon_2(d)\big] =& \mathbb{I}\big\{\big|\upsilon^{1/2}(\epsilon) - \sigma_\ell(d)\big| > \sigma_\ell(d)\varepsilon_2(d)\big\} \\ =& \mathbb{I}\big\{\big|\upsilon\big(\epsilon(d)\big) - \sigma_\ell^2(d)\big| > \sigma_\ell(d)(\upsilon^{1/2}(\epsilon) + \sigma_\ell(d))\varepsilon_2(d)\big\} \\ \leqslant& \mathbb{I}\big\{\big|\upsilon\big(\epsilon(d)\big) - \sigma_\ell^2(d)\big| > \sigma_\ell^2(d)\varepsilon_2(d)\big\}. \end{aligned}$$

Now say that from (A10) we have $\sigma_\ell^2(d) \geqslant Cd^{-r_1}$ for some $r_1 > 0$ and choose $\varepsilon_2(d) = Cd^{-r_2}$ for some arbitrary $r_2 > 0$. Then we can choose b in Theorem 47 such that $b > r_1 + r_2$ in Theorem 47 and conclude that for some $d_0 \in \mathbb{N}$, for $d \geqslant d_0$, $\mathbb{P}\big[\big|\upsilon^{1/2}\big(\epsilon(d)\big)/\sigma_\ell(d) - 1\big| > \varepsilon_2(d)\big] = 0$.

Proof. The proof relies on the decomposition in Proposition 49 and bounding of the terms $\Upsilon_{i,\epsilon}$, $i \in \{0, \ldots, 7\}$. Bounds on $\Upsilon_{1,\epsilon}$ and $\Upsilon_{2,\epsilon}$ are given in Lemma 50 and Lemma 51. Bounds on $\Upsilon_{3,\epsilon}$ and $\Upsilon_{5,\epsilon}$ are given in Lemma 52. Bounds on $\Upsilon_{4,\epsilon}$ and $\Upsilon_{6,\epsilon}$ are given in Lemma 53. Bounds on $\Upsilon_{0,\epsilon}$ and $\Upsilon_{7,\epsilon}$ are given in Lemma 54. By inspection we notice that under our assumptions, with $\iota > 1/3$ in Lemma 51 and $\zeta \in (0,1)$ in Lemma 53, each of this term is upperbounded by the product of a polynomial in the quantities defined in (A10) only, times a positive power of $\epsilon(d)$. Consequently there exist $C, r_1, r_2 > 0$, such that

$$\max_{i\in\{0,\ldots,7\}} \big|\Upsilon_{i,\epsilon(d)}\big| \leqslant Cd^{r_1}\epsilon^{r_2}(d).$$

Consequently, by choosing a_0 such that $a_0 r_2 \geqslant (r_1 + b)$ we conclude that for $\epsilon(d) = Cd^{-a}$ and $a \geqslant a_0$

$$\limsup_{d\to\infty} \max_{i\in\{0,\ldots,7\}} d^b \big|\Upsilon_{i,\epsilon(d)}\big| < \infty,$$

and we conclude. □

Proposition 49. *For any $\ell \geqslant 0$ and $\epsilon > 0$ such that $n \geqslant 2$ one has $\upsilon(\epsilon) - \sigma_\ell^2 = \sum_{i=0}^{7} \Upsilon_{i,\epsilon}$ with*

$$\begin{aligned}
\Upsilon_{0,\epsilon} :=& \epsilon^{-1} h^2 \sum_{k=1}^{n-2} \pi_{kh}(f_{kh}) \mathbb{E}\left(2\gamma_{k,\epsilon}(X_{kh}^\epsilon) - f_{kh}(X_{kh}^\epsilon)\right), \\
\Upsilon_{1,\epsilon} :=& 2\epsilon^{-1} h^2 \sum_{k=1}^{n-2} \mathbb{E}\left(\bar{f}_{kh}(X_{kh}^\epsilon)\left\{\gamma_{k,\epsilon}(X_{kh}^\epsilon) - \eta_{k,\epsilon}(X_{kh}^\epsilon)\right\}\right), \\
\Upsilon_{2,\epsilon} :=& 2h \sum_{k=1}^{n-2} \mathbb{E}\left(\bar{f}_{kh}(X_{kh}^\epsilon)\left\{\epsilon^{-1} h \eta_{k,\epsilon}(X_{kh}^\epsilon) - g_{kh}(X_{kh}^\epsilon)\right\}\right), \\
\Upsilon_{3,\epsilon} :=& 2h \sum_{k=1}^{n-2} \mathbb{E}\left(\bar{f}_{kh}(X_{kh}^\epsilon) g_{kh}(X_{kh}^\epsilon)\right) - \pi_{kh}\left(\bar{f}_{kh} g_{kh}\right), \\
\Upsilon_{4,\epsilon} :=& 2h \Big\{ \sum_{k=1}^{n-2} \pi_{kh}\left(\bar{f}_{kh} g_{kh}\right) \Big\} - 2\int_0^1 \pi_s(\bar{f}_s g_s) \mathrm{d}s, \\
\Upsilon_{5,\epsilon} :=& -\epsilon^{-1} h^2 \sum_{k=1}^{n-2} \mathbb{E}\left(\bar{f}_{kh}(X_{kh}^\epsilon) f_{kh}(X_{kh}^\epsilon)\right) - \mathrm{var}_{\pi_{kh}}\left(f_{kh}\right), \\
\Upsilon_{6,\epsilon} :=& -\epsilon^{-1} h^2 \Big\{ \sum_{k=1}^{n-2} \mathrm{var}_{\pi_{kh}}\left(f_{kh}\right) \Big\} + \ell \int_0^1 \mathrm{var}_{\pi_s}\left(f_s\right) \mathrm{d}s, \\
\Upsilon_{7,\epsilon} :=& \epsilon^{-1} h^2 \mathbb{E}\left(f_{(n-1)h}^2(X_{(n-1)h}^\epsilon) - \left[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\right]^2\right).
\end{aligned}$$

Proof. For notational simplicity we drop ϵ from $P_{s,t}^\epsilon$ here. For $n \geqslant 2$, noting that $\xi_{0,\epsilon} = 0$,

$$\begin{aligned}
\upsilon(\epsilon) &= \epsilon^{-1} h^2 \sum_{k=1}^{n-1} \mathbb{E}\left[\gamma_{k,\epsilon}^2(X_{kh}^\epsilon) - \left[P_{(k-1)h,kh}\gamma_{k,\epsilon}(X_{(k-1)h}^\epsilon)\right]^2\right] \\
&= \epsilon^{-1} h^2 \mathbb{E}\left[\gamma_{n-1,\epsilon}^2(X_{(n-1)h}^\epsilon) - \left[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\right]^2 + \sum_{k=1}^{n-2} f_{kh}(X_{kh}^\epsilon)\left\{\gamma_{k,\epsilon}(X_{kh}^\epsilon) + P_{kh,(k+1)h}\gamma_{k+1,\epsilon}(X_{kh}^\epsilon)\right\}\right] \\
&= \epsilon^{-1} h^2 \mathbb{E}\left[f_{(n-1)h}^2(X_{(n-1)h}^\epsilon) - \left[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\right]^2\right] + \epsilon^{-1} h^2 \sum_{k=1}^{n-2} \mathbb{E}\left[f_{kh}(X_{kh}^\epsilon)\left\{2\gamma_{k,\epsilon}(X_{kh}^\epsilon) - f_{kh}(X_{kh}^\epsilon)\right\}\right],
\end{aligned}$$

where the second line follows from the fact that with $W_{0,\epsilon} = \gamma_{n-1,\epsilon}^2(X_{(n-1)h}^\epsilon) - \left[P_{(n-2)h,(n-1)h}\gamma_{n-1,\epsilon}(X_{(n-2)\epsilon}^\epsilon)\right]^2$, $W_{1,\epsilon} = \sum_{k=1}^{n-2} \gamma_{k,\epsilon}^2(X_{kh}^\epsilon) - \left[P_{kh,(k+1)h}\gamma_{k,\epsilon}(X_{kh}^\epsilon)\right]^2$ and

$$\begin{aligned}
W_{2,\epsilon} &= \sum_{k=1}^{n-2} \left[P_{kh,(k+1)h}\gamma_{k,\epsilon}(X_{kh}^\epsilon)\right]^2 - \left[P_{(k-1)h,kh}\gamma_{k,\epsilon}(X_{(k-1)h}^\epsilon)\right]^2 \\
&= \left[P_{(n-2)h,(n-1)h}\gamma_{n-2,\epsilon}(X_{(n-2)h}^\epsilon)\right]^2 - \left[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\right]^2,
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{k=1}^{n-1} \gamma_{k,\epsilon}^2(X_{kh}^\epsilon) - \left[P_{(k-1)h,kh}\gamma_{k,\epsilon}(X_{(k-1)h}^\epsilon)\right]^2 &= W_{0,\epsilon} + W_{1,\epsilon} + W_{2,\epsilon} \\
&= \gamma_{n-1,\epsilon}^2(X_{(n-1)h}^\epsilon) - \left[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\right]^2 + W_{1,\epsilon},
\end{aligned}$$

and the fact that by definition $f_{kh}(x) = \gamma_{k,\epsilon}(x) - P_{kh,(k+1)h}\gamma_{k+1,\epsilon}(x)$, which is also used on the third line. $\square$

In order to control $\Upsilon_{1,\epsilon}$ and $\Upsilon_{2,\epsilon}$ we show that $\eta_{k,\epsilon}$ approximates $\gamma_{k,\epsilon}$ in Lemma 51 and that $\eta_{k,\epsilon}$ can be approximated by g_{kh} in Lemma 50.

Lemma 50. *Let $p \geqslant 1$. Assume that μ_0 satisfies (57) for some $K_{\mu_0} > 0$ and that $h(\epsilon)\epsilon^{-1} = O(1)$. Then there exists $C > 0$ such that for any $f \in C^p_{1,2}([0,1] \times \mathbb{R}^d)$*

1. *for $\ell = 0$ and any $\daleth > 1$, defining*

$$A_1 := C\alpha_{2p+1/2}\left\{L\tilde{\alpha}_{p+1/2} + \tilde{\alpha}_p\right\} \cdot \|\!|f|\!\|_p^2 \cdot \mu_0\big(\bar{V}^{(2p+1/2)}\big).$$
$$A_2 := C\alpha_{2p}K^{-1}\big\{1+\daleth\big\}\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\|\!|f|\!\|_p^2 \cdot \mu_0\big(\bar{V}^{(2p)}\big)^2,$$

then for any $\epsilon > 0$ satisfying $1/\daleth \leqslant 1 - Kh(\epsilon)\epsilon^{-1}/2$

$$\big|\Upsilon_{2,\epsilon}\big| \leqslant \big[A_2 + A_1\big(\lceil -\log(h(\epsilon)\epsilon^{-1})\rceil/K\big)^2\big]h(\epsilon)\epsilon^{-1},$$

2. *for $\ell > 0$ and $\epsilon > 0$*

$$\big|\Upsilon_{2,\epsilon}\big| \leqslant C\ell^2\mu_0\big(\bar{V}^{(p)}\big)^2\|\!|f|\!\|_p^2\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\frac{\exp\Big(-Kn(\epsilon)\ell\Big)}{1-\exp\big(-K\ell\big)}.$$

Proof. Consider first the case $\ell = 0$. Let $m(\cdot) : \mathbb{R}_+ \to \mathbb{N}$ be such that $\lim_{\epsilon\to 0} m(\epsilon)h(\epsilon)\epsilon^{-1} = \infty$ and for $s \in [0,1]$

$$I_s(\epsilon, x) := \int_0^{m(\epsilon)h(\epsilon)\epsilon^{-1}} Q^s_t\bar{f}_s(x)\mathrm{d}t,$$

with the convention that $I_s(0,x) := \lim_{\epsilon\to 0} I_s(\epsilon,x)$ (which exists, by absolute summability). Then for $k \in \{0,\ldots,n-1\}$

$$\mathbb{E}\left[\bar{f}_{kh}(X^\epsilon_{kh})\big(\epsilon^{-1}h(\epsilon)\eta_{k,\epsilon}(X^\epsilon_{kh}) - g_{kh}(X^\epsilon_{kh})\big)\right] = \mathbb{E}\left[\bar{f}_{kh}(X^\epsilon_{kh})\big(R_1(\epsilon,X^\epsilon_{kh}) + R_2(\epsilon,X^\epsilon_{kh}) + R_3(\epsilon,X^\epsilon_{kh})\big)\right]$$

where

$$\begin{aligned}
R_1(\epsilon,x) :=& h(\epsilon)\epsilon^{-1}\left(\sum_{i=0}^{m(\epsilon)-1} Q^{kh}_{ih\epsilon^{-1}}\bar{f}_{kh}(x)\right) - I_{kh}(\epsilon,x),\\
R_2(\epsilon,x) :=& h(\epsilon)\epsilon^{-1}\sum_{i=m(\epsilon)}^{n(\epsilon)-1} Q^{kh}_{ih\epsilon^{-1}}\bar{f}_{kh}(x),\\
R_3(\epsilon,x) :=& I_{kh}(\epsilon,x) - I_{kh}(0,x).
\end{aligned}$$

For the term involving $R_1(\epsilon,x)$ first notice that by the classical homogeneous equivalent of Kolmogorov's equation in Proposition 16, Lemma 61, (A3) and Lemma 57 for any $s \in [0,1]$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
\Big|\partial_t Q^s_t\bar{f}_s(x)\Big| &= \big|Q^s_t\mathcal{L}_s\bar{f}_s(x)\big|\\
&\leqslant Q^s_t\Big(\big|\big\langle\nabla U_s, \nabla\bar{f}_s\big\rangle\big| + \|\Delta\bar{f}_s\|\Big)(x),\\
&\leqslant Q^s_t\Big(\|\nabla U_s\|\cdot\|\nabla\bar{f}_s\| + \|\Delta\bar{f}_s\|\Big)(x),\\
&\leqslant L\cdot\|\nabla\bar{f}\|_p Q^s_t\Big(\bar{V}^{(1/2)}\bar{V}^{(p)}\Big)(x) + \|\Delta\bar{f}\|_p Q^s_t\Big(\bar{V}^{(p)}\Big)(x),\\
&\leqslant C\tilde{\alpha}_{p+1/2}L\cdot\|\nabla\bar{f}\|_p\bar{V}^{(p+1/2)}(x) + C\tilde{\alpha}_p\|\Delta\bar{f}\|_p\bar{V}^{(p)}(x),\\
&\leqslant C\left\{L\tilde{\alpha}_{p+1/2} + \tilde{\alpha}_p\right\}\|\!|f|\!\|_p\bar{V}^{(p+1/2)}(x).
\end{aligned}$$

Let $M(x) := \sup_{(s,t)\in[0,1]\times\mathbb{R}_+}\Big|\partial_t Q^s_t\bar{f}_s(x)\Big|$ (which can be upper bounded with the above), then we know that the difference between the Riemann sum with step-size $h(\epsilon)\epsilon^{-1}$ and its integral on the interval $[0, m(\epsilon)h(\epsilon)\epsilon^{-1}]$ yields

$$\big|R_1(\epsilon,x)\big| \leqslant M(x)h(\epsilon)\epsilon^{-1}\big(m(\epsilon)h(\epsilon)\epsilon^{-1}\big)^2,$$

leading to

$$\begin{aligned}
&\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})R_1(\epsilon, X^{\epsilon}_{kh})\right]\right| \\
&\qquad\leqslant C\left\{L\tilde{\alpha}_{p+1/2}+\tilde{\alpha}_p\right\}\|f\|_p\cdot\sup_{s\in[0,1]}\mu_s\big(|\bar{f}_s|\bar{V}^{(p+1/2)}\big)\cdot h(\epsilon)\epsilon^{-1}\big(m(\epsilon)h(\epsilon)\epsilon^{-1}\big)^2, \\
&\qquad\leqslant A_1\cdot h(\epsilon)\epsilon^{-1}\big(m(\epsilon)h(\epsilon)\epsilon^{-1}\big)^2.
\end{aligned}$$

where

$$A_1 := C\alpha_{2p+1/2}\left\{L\tilde{\alpha}_{p+1/2}+\tilde{\alpha}_p\right\}\cdot\|f\|_p^2\cdot\mu_0\big(\bar{V}^{(2p+1/2)}\big).$$

We define and upper bound the following quantities,

$$\begin{aligned}
R_{2,1} :=& h(\epsilon)\epsilon^{-1}\sum_{i=m(\epsilon)}^{n(\epsilon)-1}\mathrm{var}_{\mu^{\epsilon}_{kh}}\left[Q^{s}_{ih\epsilon^{-1}}\bar{f}_{kh}\right]^{1/2} \\
&\leqslant \frac{1}{K}\frac{\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)}{\big[1-\exp\big(-Kh(\epsilon)\epsilon^{-1}\big)\big]/(Kh(\epsilon)\epsilon^{-1})}\sup_{(s,t)\in[0,1]\times\mathbb{R}_+}\mathrm{var}_{\mu^{\epsilon}_s Q^s_t}\big[f_s\big]^{1/2}, \\
R_{2,2} :=& h(\epsilon)\epsilon^{-1}\sum_{i=m(\epsilon)}^{n(\epsilon)-1}\left|\mathbb{E}\left[Q^{s}_{ih\epsilon^{-1}}\bar{f}_{kh}(X^{\epsilon}_{kh})\right]\right| \\
&\leqslant \frac{\tilde{\alpha}_p}{K}\frac{\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)}{\big[1-\exp\big(-Kh(\epsilon)\epsilon^{-1}\big)\big]/(Kh(\epsilon)\epsilon^{-1})}\|f\|_p\sup_{(s,t)\in[0,1]\times\mathbb{R}_+}\mu^{\epsilon}_t\big[W^{(p)}(\delta_{\cdot},\pi_s)\big], \\
R_{3,1} :=& \int_{m(\epsilon)h\epsilon^{-1}}^{\infty}\mathrm{var}_{\mu^{\epsilon}_{kh}}\big[Q^{kh}_t\bar{f}_{kh}\big]^{1/2}\mathrm{d}t \leqslant \frac{1}{K}\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)\sup_{(s,t)\in[0,1]\times\mathbb{R}_+}\mathrm{var}_{\mu^{\epsilon}_s Q^s_t}\left[f_s\right]^{1/2}, \\
R_{3,2} :=& \int_{m(\epsilon)h\epsilon^{-1}}^{\infty}\big|\mathbb{E}\big[Q^s_t\bar{f}_s(X^{\epsilon}_s)\big]\big|\mathrm{d}t \leqslant \frac{\tilde{\alpha}_p}{K}\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)\cdot\|f\|_p\sup_{(s,t)\in[0,1]\times\mathbb{R}_+}\mu^{\epsilon}_t\big[W^{(p)}(\delta_{\cdot},\pi_s)\big],
\end{aligned}$$

where the upper bounds follow from the homogeneous equivalent of Lemma 23, (102) and Jensen's inequality. We now apply successively the Cauchy-Schwarz and Minkowski inequalities (the latter in its sum and integral form), and note the standard inequality $\mathbb{E}\big[Z^2\big]^{1/2}\leqslant\mathrm{var}\big[Z\big]^{1/2}+\big|\mathbb{E}\big[Z\big]\big|$ for any random variable Z

$$\begin{aligned}
\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})[R_2(\epsilon,X^{\epsilon}_{kh})+R_3(\epsilon,X^{\epsilon}_{kh})]\right]\right| &\leqslant \mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})^2\right]^{1/2}\mathbb{E}\left[\mathbb{E}[R_2(\epsilon,X^{\epsilon}_{kh})+R_3(\epsilon,X^{\epsilon}_{kh})\mid\mathcal{F}_{kh}]^2\right]^{1/2} \\
&\leqslant \mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})^2\right]^{1/2}\Big\{\mathbb{E}\left[\mathbb{E}[R_2(\epsilon,X^{\epsilon}_{kh})\mid\mathcal{F}_{kh}]^2\right]^{1/2}+\mathbb{E}\left[\mathbb{E}[R_3(\epsilon,X^{\epsilon}_{kh})\mid\mathcal{F}_{kh}]^2\right]^{1/2}\Big\}, \quad (97)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}[R_2(\epsilon,X^{\epsilon}_{kh})\mid\mathcal{F}_{kh}]^2\right]^{1/2} &\leqslant h(\epsilon)\epsilon^{-1}\sum_{i=m(\epsilon)}^{n(\epsilon)-1}\mathbb{E}\left[\big(Q^{kh}_{ih\epsilon^{-1}}\bar{f}_{kh}(X^{\epsilon}_{kh})\big)^2\right]^{1/2} \\
&\leqslant R_{2,1}+R_{2,2},
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}[R_3(\epsilon,X^{\epsilon}_{kh})\mid\mathcal{F}_{kh}]^2\right]^{1/2} &\leqslant \int_{m(\epsilon)h(\epsilon)\epsilon^{-1}}^{\infty}\mathbb{E}\left[\big(Q^{kh}_t\bar{f}_{kh}(X^{\epsilon}_{kh})\big)^2\right]^{1/2}\mathrm{d}t \\
&\leqslant R_{3,1}+R_{3,2}.
\end{aligned}$$

Note that from Lemmas 56, 14 and 58,

$$\begin{aligned}
\mu^{\epsilon}_t\big[W^{(p)}(\delta_{\cdot},\pi_s)\big] &\leqslant C\mu^{\epsilon}_t\bar{V}^{(p+1/2)}\cdot\pi_s\bar{V}^{(p+1/2)} \\
&\leqslant C\alpha_{p+1/2}\mu_0\bar{V}^{(p+1/2)}\cdot\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)},
\end{aligned}$$

and together with Lemma 58 we deduce that

$$\sum_{i=1}^{2} \left|R_{2,i}\right| + \left|R_{3,i}\right| \leqslant C\left[1 + \frac{Kh(\epsilon)\epsilon^{-1}}{1-\exp\left(-Kh(\epsilon)\epsilon^{-1}\right)}\right]\Big\{\|\nabla f\|_p\big[K^{-1}+K_{\mu_0}^{-1}\big]^{1/2}\big(\tilde{\alpha}_{2p}\alpha_{2p}\mu_0\bar{V}^{(2p)}\big)^{1/2}$$
$$+\tilde{\alpha}_p\alpha_{p+1/2}\|f\|_p\mu_0\bar{V}^{(p+1/2)}\cdot\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)}\Big\}K^{-1}\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)$$
$$\leqslant \bar{A}_2\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)$$

where the last inequality holds for $1/\beth < 1 - Kh(\epsilon)\epsilon^{-1}/2$, thanks to Lemma 60, and

$$\bar{A}_2 := CK^{-1}\{1+\beth\}\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\|\!|f|\!\|\mu_0\bar{V}^{(2p)}.$$

Together with (97) we deduce that for $1/\beth < 1 - Kh(\epsilon)\epsilon^{-1}/2$

$$\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})\big(\epsilon^{-1}h(\epsilon)\eta_{k,\epsilon}(X^{\epsilon}_{kh}) - g_{kh}(X^{\epsilon}_{kh})\big)\right]\right| \leqslant A_1 h(\epsilon)\epsilon^{-1}\big(m(\epsilon)h(\epsilon)\epsilon^{-1}\big)^2 + A_2\exp\Big(-Km(\epsilon)h(\epsilon)\epsilon^{-1}\Big)$$

with $A_2 := C\bar{A}_2\cdot\|f\|_p\alpha_{2p}\mu_0\bar{V}^{(2p)}$ and by taking $Km(\epsilon)h(\epsilon)\epsilon^{-1} = \lceil -\log(h(\epsilon)\epsilon^{-1})\rceil$ we obtain

$$h(\epsilon)\epsilon^{-1}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})\big(\eta_{k,n}(X^{\epsilon}_{kh}) - g_{kh}(X^{\epsilon}_{kh})\big)\right]\right| \leqslant h(\epsilon)\epsilon^{-1}\big[A_2 + A_1\big(\lceil -\log(h(\epsilon)\epsilon^{-1})\rceil/K\big)^2\big].$$

The scenario $\ell > 0$ is more direct and can be bounded in a similar way to the term dependent on R_2 above–as a result for $k \in \{0,\ldots,n-1\}$

$$\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})\big(\ell\eta_{k,\epsilon}(X^{\epsilon}_{kh}) - g_{kh}(X^{\epsilon}_{kh})\big)\right]\right| = \ell\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})\sum_{i=n(\epsilon)}^{\infty}Q^{kh}_{i\ell}\bar{f}_{kh}(X^{\epsilon}_{kh})\right]\right|$$
$$\leqslant C\ell^2\mu_0\big(\bar{V}^{(p)}\big)^2\|\!|f|\!\|_p^2\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\frac{\exp\Big(-Kn(\epsilon)\ell\Big)}{1-\exp\big(-K\ell\big)}.$$

□

Lemma 51. *Let $p \geqslant 1$, $f \in C^p_{1,2}([0,1]\times\mathbb{R}^d)$, $\iota \in \big(0,1\big)$, define for any $\epsilon > 0$ and $k \in \{0,\ldots,n-1\}$ $\tau_{k,\epsilon} := (kh + \daleth h^{\iota}) \wedge 1$ for some $\daleth > 0$, and define for $k \in \{0,\ldots,n-1\}$ and $x \in \mathbb{R}^d$*

$$T_{1,k,\epsilon} := \sum_{i=k}^{\lfloor\tau_{k,\epsilon}h^{-1}\rfloor-1} P^{\epsilon}_{kh,ih}f_{ih}(x) - Q^{kh,\epsilon}_{(i-k)h}f_{ih}(x), \qquad T_{2,k,\epsilon} := \sum_{i=k}^{\lfloor\tau_{k,\epsilon}h^{-1}\rfloor-1} Q^{kh,\epsilon}_{(i-k)h}f_{ih}(x) - Q^{kh,\epsilon}_{(i-k)h}f_{kh}(x),$$

$$T_{3,k,\epsilon} := (\lfloor\tau_{k,\epsilon}h^{-1}\rfloor - k)\pi_{kh}(f_{kh}), \qquad T_{4,k,\epsilon} := \sum_{i=\lfloor\tau_{k,\epsilon}h^{-1}\rfloor}^{n-1} P^{\epsilon}_{kh,ih}f_{ih}(x) - Q^{kh,\epsilon}_{(i-k)h}\bar{f}_{kh}(x),$$

with the standard conventions that that $T_{1,k,\epsilon} = T_{2,k,\epsilon} = T_{3,k,\epsilon} = 0$ when $\lfloor\tau_{k,\epsilon}h^{-1}\rfloor = k$ and $T_{4,k,\epsilon} = 0$ when $\lfloor\tau_{k,\epsilon}h^{-1}\rfloor = n$. Then

$$\left|\Upsilon_{1,\epsilon}\right| \leqslant 2\epsilon^{-1}h^2\sum_{k=1}^{n-1}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{3,k,\epsilon}\right]\right| + 2\epsilon^{-1}h\sum_{i=1,i\neq3}^{4}\max_{k\in\{0,\ldots,n-1\}}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{i,k,\epsilon}\right]\right|,$$

and there exists $C > 0$ such that for any $\epsilon > 0$ and $\ell \geqslant 0$,

$$\max_{k\in\{0,\ldots,n-1\}}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{1,k,\epsilon}\right]\right| \leqslant C\daleth^3\alpha_p\tilde{\alpha}_{p+1/2}\alpha_{2p+1/2}M\cdot |||\bar{f}|||_p^2\cdot \sup_{s\in[0,1]}\bar{V}(x^{\star}_s)^{1/2}\cdot\mu_0\left(\bar{V}^{(2p+1/2)}\right)\cdot\epsilon^{-1}h^{3\iota},$$

$$\max_{k\in\{0,\ldots,n-1\}}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{2,k,\epsilon}\right]\right| \leqslant C\daleth^2\tilde{\alpha}_p\alpha_{2p}|||f|||_p^2\mu_0\left(\bar{V}^{(2p)}\right)h^{2\iota},$$

$$2\epsilon^{-1}h^2\sum_{k=0}^{n-1}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{3,k,\epsilon}\right]\right| \leqslant C\daleth\left\{|||f|||_p\sup_{s\in[0,1]}\pi_s\bar{V}^{(2[p\vee p_0]+1/2)}\Big[\frac{\|\nabla\phi\|_{p_0}}{K^2}+\alpha_p\mu_0\bar{V}^{(p+1/2)}\Big]\right\}^2$$
$$\times\Big\{-h\ln(\epsilon)/K+\epsilon^{-1}h^2+\epsilon h^{\iota}\Big\}.$$

Define

$$A := \gimel\alpha_{2p}|||f|||_p^2\mu_0V^{(2p)}\left[K^{-1}+K_{\mu_0}^{-1}\right]^{1/2}$$
$$+\frac{\gimel}{K}\alpha_{2p}\mu_0\big(\bar{V}^{(2p)}\big)^2|||f|||_p^2\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)}+\big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}$$

then there exists $C > 0$ such that for any $\gimel > 1$ and $\gimel^{-1} < 1 - Kh\epsilon^{-1}/2$

$$\max_{k\in\{0,\ldots,n-1\}}\left|\mathbb{E}\left[\bar{f}_{kh}(X^{\epsilon}_{kh})T_{4,k,\epsilon}\right]\right| \leqslant C\cdot A\exp\Big(-K[\daleth h^{\iota-1}-1]h\epsilon^{-1}\Big)\cdot\big[(\epsilon h^{-1})\vee 1\big].$$

Proof. For the first statement, simply notice that for any $k\in\{0,\ldots,n-1\}$

$$\gamma_{k,\epsilon}(x)-\eta_{k,\epsilon}(x) = T_{1,k,\epsilon}+T_{2,k,\epsilon}+T_{3,k,\epsilon}+T_{4,k,\epsilon}.$$

From Proposition 16 (and its time-homogeneous version) we deduce that for $0\leqslant s<u\leqslant 1$

$$\begin{aligned}
Q^{s,\epsilon}_{u-s}f_u(x)-P^{\epsilon}_{s,u}f_u(x) &= \int_0^{u-s}\frac{\partial}{\partial t}Q^{s,\epsilon}_tP^{\epsilon}_{s+t,u}f_u(x)\mathrm{d}t\\
&= \int_0^{u-s}Q^{s,\epsilon}_t\left(\frac{\partial}{\partial t}P^{\epsilon}_{s+t,u}f_u+\mathcal{L}_sP^{\epsilon}_{s+t,u}f_u\right)(x)\mathrm{d}t\\
&= \int_0^{u-s}Q^{s,\epsilon}_t\left(\mathcal{L}_s-\mathcal{L}_{s+t}\right)P^{\epsilon}_{s+t,u}f_u(x)\mathrm{d}t\\
&= -\epsilon^{-1}\int_0^{u-s}Q^{s,\epsilon}_t\left(\big\langle\nabla U_s-\nabla U_{s+t},\nabla P^{\epsilon}_{s+t,u}f_u\big\rangle\right)(x)\mathrm{d}t.
\end{aligned}$$

Now by application of the Cauchy-Schwarz inequality, Lemma 18 and (A5), we deduce that

$$\begin{aligned}
\big|Q^{s,\epsilon}_{u-s}f_u(x)-P^{\epsilon}_{s,u}f_u(x)\big| &\leqslant \epsilon^{-1}M\cdot\int_0^{u-s}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot P^{\epsilon}_{s+t,u}\|\nabla f_u\|\right)(x)\cdot t\exp\big(-K\epsilon^{-1}(u-s-t)\big)\mathrm{d}t\\
&\leqslant \epsilon^{-1}M\sup_{t\in[0,u-s]}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot P^{\epsilon}_{s+t,u}\|\nabla f_u\|\right)(x)\cdot\frac{1}{2}(u-s)^2\\
&\leqslant C\epsilon^{-1}M(u-s)^2\sup_{t\in[0,u-s]}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot P^{\epsilon}_{s+t,u}\|\nabla f_u\|\right)(x).
\end{aligned}$$

Further by assumption $\|\nabla f\|_p<\infty$ and from Lemma 14

$$\begin{aligned}
\sup_{t\in[0,u-s]}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot P^{\epsilon}_{s+t,u}\|\nabla f_u\|\right)(x) &\leqslant \|\nabla f\|_p\cdot\sup_{t\in[0,u-s]}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot P^{\epsilon}_{s+t,u}\bar{V}^{(p)}\right)(x)\\
&\leqslant \alpha_p\|\nabla f\|_p\cdot\sup_{t\in[0,u-s]}Q^{s,\epsilon}_t\left(\sqrt{\bar{V}_s}\cdot\bar{V}^{(p)}\right)(x).
\end{aligned}$$

Now from Proposition 57 and from Lemma 61, for $s,t \in [0,1]$ and $\epsilon > 0$

$$Q_t^{s,\epsilon}\left(\sqrt{\bar{V}_s}\cdot \bar{V}^{(p)}\right)(x) \leqslant C\tilde{\alpha}_{p+1/2}\sqrt{\bar{V}(x_s^\star)}\cdot \bar{V}^{(p+1/2)}(x).$$

We also know that

$$\begin{aligned}\mu_s\left(\left|\bar{f}_s\right|\bar{V}^{(p+1/2)}\right) &\leqslant \|\bar{f}\|_p\cdot \mu_s\left(\bar{V}^{(p)}\bar{V}^{(p+1/2)}\right)\\ &\leqslant C\|\bar{f}\|_p\cdot \mu_s\left(\bar{V}^{(2p+1/2)}\right)\\ &\leqslant C\alpha_{2p+1/2}\|\bar{f}\|_p\cdot \mu_0\left(\bar{V}^{(2p+1/2)}\right),\end{aligned}$$

where we have used Lemma 57 and Lemma 14. Since $u\mapsto u-kh$ is non-decreasing, non-negative for $u\geqslant kh$ and $\lfloor \tau_{k,\epsilon}h^{-1}\rfloor h \leqslant \tau_{k,\epsilon}$

$$\begin{aligned}\left|T_{1,k,\epsilon}\right| &\leqslant C\alpha_p\epsilon^{-1}M\|\nabla f\|_p\cdot \sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sqrt{\bar{V}_s}\cdot \bar{V}^{(p)}\right)(x)\int_{kh}^{\tau_{k,\epsilon}}(u-kh)^2\mathrm{d}u\\ &= C\alpha_p\daleth^3 M\|\nabla f\|_p\cdot \sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sqrt{\bar{V}_s}\cdot \bar{V}^{(p)}\right)(x)\cdot \epsilon^{-1}h^{3\iota},\end{aligned}$$

and with the bounds on $\sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sqrt{\bar{V}_s}\cdot \bar{V}^{(p)}\right)(x)$ and $\mu_s\left(\left|\bar{f}_s\right|\bar{V}^{(p+1/2)}\right)$ we obtain

$$\begin{aligned}&\max_{k\in\{0,\dots,n-1\}}\mathbb{E}\left[\left|\bar{f}_{kh}(X_{kh}^\epsilon)T_{1,k,\epsilon}\right|\right] \leqslant\\ &\qquad C\daleth^3\alpha_p\tilde{\alpha}_{p+1/2}\alpha_{2p+1/2}M\cdot |\!|\!|\bar{f}|\!|\!|_p^2\cdot \sup_{s\in[0,1]}\bar{V}(x_s^\star)^{1/2}\cdot \mu_0\left(\bar{V}^{(2p+1/2)}\right)\cdot \epsilon^{-1}h^{3\iota}.\end{aligned}$$

For the term $T_{2,k,\epsilon}$ we use the smoothness $s\mapsto f_s(x)$ and its derivative, the fact that $i\mapsto i-k$ is non-decreasing and non-negative for $i\geqslant k$ and again the fact that $\lfloor \tau_{k,\epsilon}h^{-1}\rfloor h\leqslant \tau_{k,\epsilon}$

$$\begin{aligned}\left|T_{2,k,\epsilon}\right| &\leqslant \sum_{i=k}^{\lfloor \tau_{k,\epsilon}h^{-1}\rfloor -1} Q_{(i-k)h}^{kh,\epsilon}\left(\left|f_{ih}-f_{kh}\right|\right)(x)\\ &\leqslant \sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sup_{u\in[0,1]}\left|\partial_t f_u(\cdot)\right|\right)(x)\cdot \sum_{i=k}^{\lfloor \tau_{k,\epsilon}h^{-1}\rfloor -1} ih-kh\\ &\leqslant \sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sup_{u\in[0,1]}\left|\partial_t f_u(\cdot)\right|\right)(x)\cdot\int_{kh}^{\tau_{k,\epsilon}}(u-kh)\mathrm{d}u\\ &= \frac{\daleth^2}{2}\sup_{s,t\in[0,1]} Q_t^{s,\epsilon}\left(\sup_{u\in[0,1]}\left|\partial_t f_u(\cdot)\right|\right)(x)\cdot h^{2\iota}.\end{aligned}$$

Now by assumption $\|\partial_t f\|_p < \infty$ and from Lemma 61, for $s,t\in[0,1]$ and $\epsilon>0$

$$\begin{aligned}Q_t^{s,\epsilon}\left(\sup_{u\in[0,1]}\left|\partial_t f_u(\cdot)\right|\right)(x) &\leqslant \|\partial_t f\|_p Q_t^{s,\epsilon}\bar{V}^{(p)}(x)\\ &\leqslant \tilde{\alpha}_p\|\partial_t f\|_p\bar{V}^{(p)}(x).\end{aligned}$$

Therefore

$$\begin{aligned}\max_{k\in\{0,\dots,n-1\}}\mathbb{E}\left[\left|\bar{f}_{kh}(X_{kh}^\epsilon)T_{2,k,\epsilon}\right|\right] &\leqslant C\daleth^2\tilde{\alpha}_p h^{2\iota}\|\partial_t f\|_p\sup_{s\in[0,1]}\mu_s\left(\left|\bar{f}_s\right|\bar{V}^{(p)}\right).\\ &\leqslant C\daleth^2\tilde{\alpha}_p\alpha_{2p}\|\partial_t f\|_p\|f\|_p\mu_0\left(\bar{V}^{(2p)}\right)h^{2\iota}.\\ &\leqslant C\daleth^2\tilde{\alpha}_p\alpha_{2p}|\!|\!|f|\!|\!|_p^2\mu_0\left(\bar{V}^{(2p)}\right)h^{2\iota}.\end{aligned}$$

For $T_{3,k,\epsilon}$ we exploit that for $s \in [0,1]$, $\mu_s^\epsilon(f_s) = 0$

$$\left|\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)T_{3,k,\epsilon}\right]\right| = (\lfloor \tau_{k,\epsilon}h^{-1}\rfloor - k)\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)\right]^2$$

and therefore from Lemma 58 and the fact that $\lfloor \tau_{k,\epsilon}h^{-1}\rfloor - k \leqslant \urcorner h^{\iota-1}$we deduce that for $k \in \{0,\ldots,n-1\}$

$$\begin{aligned}&\left|\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)T_{3,k,\epsilon}\right]\right| \\ &\qquad \leqslant C\urcorner\left\{\|\!|f|\!\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p\vee p_0]+1/2)}\Big[\frac{\|\nabla\phi\|_{p_0}}{K^2}\epsilon + \alpha_p\mu_0\bar{V}^{(p+1/2)}\exp\big(-K\epsilon^{-1}hk\big)\Big]\right\}^2 h^{\iota-1}.\end{aligned}$$

and in particular for $k \geqslant \lceil -\ln(\epsilon)/(K\epsilon^{-1}h)\rceil$ and letting

$$B := \urcorner\left\{\|\!|f|\!\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p\vee p_0]+1/2)}\Big[\frac{\|\nabla\phi\|_{p_0}}{K^2} + \alpha_p\mu_0\bar{V}^{(p+1/2)}\Big]\right\}^2$$

we have

$$\left|\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)T_{3,k,\epsilon}\right]\right| \leqslant CB\epsilon^2 h^{\iota-1}$$

As a result

$$\begin{aligned}2\epsilon^{-1}h^2\sum_{k=1}^{n-1}\left|\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)T_{3,k,\epsilon}\right]\right| &\leqslant C\cdot B\epsilon^{-1}h\Big\{h\lceil -\ln(\epsilon)/(K\epsilon^{-1}h)\rceil + \epsilon^2h^{\iota-1}\Big\} \\ &\leqslant C\cdot B\Big\{-h\ln(\epsilon)/K + \epsilon^{-1}h^2 + \epsilon h^\iota\Big\}.\end{aligned}$$

Finally, defining

$$\begin{aligned}\mathcal{T}_{4,1} &:= \sum_{i=\lfloor \tau_{k,\epsilon}h^{-1}\rfloor}^{n-1}\left|\mathbb{E}\Big[\bar{f}_{kh}(X_{kh}^\epsilon)P_{kh,ih}^\epsilon f_{ih}\big(X_{kh}^\epsilon\big)\Big]\right| \\ \mathcal{T}_{4,2} &:= \sum_{i=\lfloor \tau_{k,\epsilon}h^{-1}\rfloor}^{n-1}\left|\mathbb{E}\Big[\bar{f}_{kh}(X_{kh}^\epsilon)Q_{(i-k)h}^{kh,\epsilon}\bar{f}_{kh}\big(X_{kh}^\epsilon\big)\Big]\right|\end{aligned}$$

we have

$$\left|\mathbb{E}\left[\bar{f}_{kh}(X_{kh}^\epsilon)T_{4,k,\epsilon}\right]\right| \leqslant \mathcal{T}_{4,1} + \mathcal{T}_{4,2}.$$

The term $\mathcal{T}_{4,2}$ is bounded in the same way the R_2 dependent term in the proof of Lemma 50, yielding

$$\begin{aligned}\mathcal{T}_{4,2} &\leqslant C\alpha_{2p}\mu_0\big(\bar{V}^{(2p)}\big)^2\|\!|f|\!\|_p^2\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\frac{\exp\Big(-K\big[\lfloor \tau_{k,\epsilon}h^{-1}\rfloor - k\big]h\epsilon^{-1}\Big)}{\big[1-\exp\big(-Kh\epsilon^{-1}\big)\big]/h\epsilon^{-1}} \\ &\leqslant C\urcorner\alpha_{2p}\mu_0\big(\bar{V}^{(2p)}\big)^2\|\!|f|\!\|_p^2\Big\{\tilde{\alpha}_p\alpha_{p+1/2}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)} + \big(\tilde{\alpha}_{2p}\alpha_{2p}\big[K^{-1}+K_{\mu_0}^{-1}\big]\big)^{1/2}\Big\}\frac{\exp\Big(-K\big[\urcorner h^{\iota-1}-1\big]h\epsilon^{-1}\Big)}{K}.\end{aligned}$$

Now we note that by the Cauchy-Schwarz inequality, Lemma 23 and Lemma 58,

$$\begin{aligned}\mathcal{T}_{4,1} &\leqslant \left|\mathbb{E}\left[\bar{f}_{kh}^2(X_{kh}^\epsilon)\right]^{1/2}\sum_{i=\lfloor \tau_{k,\epsilon}h^{-1}\rfloor}^{n-1}\mathrm{var}_{\mu_{kh}^\epsilon}\left[P_{kh,ih}^\epsilon f_{ih}\right]^{1/2}\right. \\ &\leqslant C\alpha_{2p}^{1/2}\|f\|_p\mu_0\big(\bar{V}^{(2p)}\big)^{1/2}\|\nabla f\|_p\big(\alpha_{2p}\cdot\big[K^{-1}+K_{\mu_0}^{-1}\big]\mu_0\bar{V}^{(2p)}\big)^{1/2}\frac{\exp\Big(-K\epsilon^{-1}\big(\lfloor \tau_{k,\epsilon}h^{-1}\rfloor - k\big)h\Big)}{1-\exp\Big(-K\epsilon^{-1}h\Big)}. \\ &\leqslant C\urcorner\|\!|f|\!\|_p^2\mu_0\bar{V}^{(2p)}\alpha_{2p}\left[K^{-1}+K_{\mu_0}^{-1}\right]^{1/2}\frac{\exp\Big(-K\big[\urcorner h^{\iota-1}-1\big]h\epsilon^{-1}\Big)}{K}\epsilon h^{-1}.\end{aligned}$$

because $\lfloor \tau_{k,\epsilon} h^{-1} \rfloor h \geqslant \tau_{k,\epsilon} - h$.

$\square$

Lemma 52. *For any $\beth > 1$ and $\epsilon, h, K > 0$ such that $\beth^{-1} < 1 - K\epsilon^{-1}h/2$ we have for $\ell \geqslant 0$,*

$$\begin{aligned}\left|\Upsilon_{3,\epsilon}\right| \leqslant C\frac{\beth\tilde{\alpha}_p}{K}\|\!|f|\!\|_p^2 \left[\sup_{s\in[0,1]} \pi_s \bar{V}^{(2[(2p+1/2)\vee p_0]+1/2)}\right]^3 \\ \times \left\{1 + K^{-2}\|\nabla\phi\|_{p_0} + \mu_0 \bar{V}^{(2p+1)}\alpha_{2p+1/2}\left(1+\frac{\beth}{K}\right)\right\}\epsilon,\end{aligned}$$

and

$$\left|\Upsilon_{5,\epsilon}\right| \leqslant C\left(\|\!|f|\!\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[(2p)\vee p_0]+1/2)}\right)^2 \left\{1 + K^{-2}\|\nabla\phi\|_{p_0} h + \frac{\beth(\alpha_{2p}\vee\alpha_p)}{K}\mu_0 \bar{V}^{(2p+1/2)}\right\}^2 (\epsilon \vee h).$$

Proof. First we establish some intermediate results. Choose $r(0,\epsilon) := \lceil -\ln(\epsilon)/K \rceil$ and $r(\ell,\epsilon) := \lceil -\ln(\epsilon)/(K\ell) \rceil$ for $\ell > 0$, then

$$\begin{cases} e^{-K\ell r(\ell,\epsilon)}, & \ell > 0 \\ e^{-K r(\ell,\epsilon)}, & \ell = 0 \end{cases} \leqslant \epsilon.$$

From Lemma 62 this implies that for any $\ell \geqslant 0$ such that $\beth^{-1} \leqslant 1 - K\ell/2$

$$\Delta_{s,r(\ell,\epsilon)}(x) \leqslant C\beth\frac{\tilde{\alpha}_p}{K}\|f\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)} \cdot \bar{V}^{(p+1/2)}(x)\epsilon,$$

and

$$\sup_{s,r\in[0,1]\times\mathbb{N}\cup\{\infty\}} \|g_{s,r}\|_{p+1/2} \leqslant C\beth\frac{\tilde{\alpha}_p}{K}\|f\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)}.$$

From the homogeneous version of Lemma 18 and Lemma 14 we have that for any $x \in \mathbb{R}^d$

$$\|\nabla g_{s,r}(x)\| \leqslant \frac{\tilde{\alpha}_p}{K}\|\nabla f\|_p \bar{V}^{(p)}(x) \begin{cases} \frac{K\ell}{1-e^{-K\ell}}, & \ell > 0, \\ 1, & \ell = 0 \end{cases}.$$

and since $\bar{V}^{(p)}(x) \leqslant C\bar{V}^{(p+1/2)}(x)$ we deduce that for any $\ell \geqslant 0$ such that $\beth^{-1} \leqslant 1 - K\ell/2$

$$\sup_{(s,r)\in[0,1]\times\mathbb{N}} \|g_{s,r}\|_p \vee \|\nabla g_{s,r}\|_{p+1/2} \leqslant C\beth\frac{\tilde{\alpha}_p}{K}\|\nabla f\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)}.$$

Now for $r \in \mathbb{N}$

$$\left|\Upsilon_{3,\epsilon}\right| \leqslant \Upsilon_{3,\epsilon,r}^{(1)} + \Upsilon_{3,\epsilon,r}^{(2)},$$

with

$$\begin{aligned}\Upsilon_{3,\epsilon,r}^{(1)} &:= 2h\Big|\sum_{k=1}^{n-2} \mathbb{E}\left(\bar{f}_{kh}(X_{kh}^\epsilon) g_{kh,r}(X_{kh}^\epsilon)\right) - \pi_{kh,r}\left(\bar{f}_{kh} g_{kh,r}\right)\Big|, \\ \Upsilon_{3,\epsilon,r}^{(2)} &:= 2h\sum_{k=1}^{n-2} \mathbb{E}\left(\left|\bar{f}_{kh}(X_{kh}^\epsilon)\right| \Delta_{kh,r}(X_{kh}^\epsilon)\right) + \pi_{kh}\left(\left|\bar{f}_{kh}\right| \Delta_{kh,r}\right).\end{aligned}$$

Note that from above and Lemma 57 we have $\|\nabla(f g_{s,r})\|_{2p+1/2} \leqslant 4\|\nabla f\|_p \|g_{s,r}\|_{p+1/2} + 4\|f\|_p \|\nabla g_{s,r}\|_{p+1/2}$, and we deduce that for any $\ell \geqslant 0$ and $\beth^{-1} \leqslant 1 - Kh\epsilon^{-1}/2$

$$\sup_{r\in\mathbb{N}} \|\nabla(f g_r)\|_{2p+1/2} \leqslant C\beth\frac{\tilde{\alpha}_p}{K}\|\!|f|\!\|_p^2 \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)}.$$

From Lemma 58

$$\Upsilon^{(1)}_{3,\epsilon,r} \leqslant C \sup_{(s,r)\in[0,1]\times\mathbb{N}} \|\nabla(fg_{s,r})\|_{2p+1/2}\times$$
$$\times \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[(2p+1/2)\vee p_0]+1/2)}\Big\{K^{-2}\|\nabla\phi\|_{p_0} + \frac{\alpha_{2p+1/2}}{K}\mu_0 \bar{V}^{(2p+1)} \frac{1}{\big[1-\exp\big(-K\epsilon^{-1}h\big)\big]/(Kh\epsilon^{-1})}\Big\}\epsilon$$
$$\leqslant C \sup_{(s,r)\in[0,1]\times\mathbb{N}} \|\nabla(fg_{s,r})\|_{2p+1/2} \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[(2p+1/2)\vee p_0]+1/2)}\Big\{K^{-2}\|\nabla\phi\|_{p_0} + \frac{\alpha_{2p+1/2}}{K}\mu_0 \bar{V}^{(2p+1)}\daleth\Big\}\epsilon.$$

Further from the bound on $\Delta_{s,r}(\cdot)$ above

$$\Upsilon^{(2)}_{3,\epsilon,r} \leqslant C\daleth\frac{\tilde{\alpha}_p}{K}\|f\|_p^2 \sup_{s\in[0,1]} \pi_s\bar{V}^{(p+1/2)}\cdot\big(\alpha_{2p+1/2}\mu_0\bar{V}^{(2p+1/2)} + \sup_{s\in[0,1]}\pi_s\bar{V}^{(2p+1/2)}\big)\epsilon.$$

As a result

$$\big|\Upsilon_{3,\epsilon}\big| \leqslant C\daleth\frac{\tilde{\alpha}_p}{K}|||f|||_p^2 \sup_{s\in[0,1]} \pi_s\bar{V}^{(p+1/2)} \sup_{s\in[0,1]} \pi_s\bar{V}^{(2[(2p+1/2)\vee p_0]+1/2)}$$
$$\left\{\sup_{s\in[0,1]} \pi_s\bar{V}^{(2p+1/2)} + K^{-2}\|\nabla\phi\|_{p_0} + \mu_0\bar{V}^{(2p+1)}\alpha_{2p+1/2}\left(1+\frac{\daleth}{K}\right)\right\}\epsilon$$

from which we conclude. We turn to the second statement. Note that we have the slight simplification $\Upsilon_{5,\epsilon} = -\epsilon^{-1}h^2\sum_{k=1}^{n-2}\mathbb{E}\big[f^2_{kh}(X^\epsilon_{kh})\big] - \mathrm{var}_{\pi_{kh}}\big[f_{kh}\big]$, that

$$\Big|\mathbb{E}\Big[f^2_{kh}(X^\epsilon_{kh})\Big] - \mathrm{var}_{\pi_{kh}}\big[f_{kh}\big]\Big| \leqslant \Big|\mathbb{E}\Big[f^2_{kh}(X^\epsilon_{kh})\Big] - \pi_{kh}f^2_{kh}\Big| + \big|\pi_{kh}f_{kh}\big|^2$$

and $f\in C^p_{0,2}\big(\mathbb{R}^d\big)$ implies that $f^2\in C^{2p}_{0,2}\big([0,1]\times\mathbb{R}^d\big)$ from Lemma 57. Now from Lemma 58,

$$\big|\Upsilon_{5,\epsilon}\big| \leqslant C\|f\|_p\|\nabla f\|_p \sup_{s\in[0,1]}\pi_s\bar{V}^{(2[(2p)\vee p_0]+1/2)}\Big\{K^{-2}\|\nabla\phi\|_{p_0}h + \alpha_{2p}\mu_0\bar{V}^{(2p+1/2)}\frac{\exp\big(-K\epsilon^{-1}h\big)}{1-\exp\big(-K\epsilon^{-1}h\big)}\epsilon^{-1}h\Big\}h.$$
$$+ C\left(\|\nabla f\|_p\sup_{s\in[0,1]}\pi_s\bar{V}^{(2[p\vee p_0]+1/2)}\right)^2\Big\{K^{-2}\|\nabla\phi\|_{p_0}h + \alpha_p\mu_0\bar{V}^{(p+1/2)}\frac{\exp\big(-K\epsilon^{-1}h\big)}{1-\exp\big(-K\epsilon^{-1}h\big)}\epsilon^{-1}h\Big\}^2\epsilon$$

Therefore

$$\big|\Upsilon_{5,\epsilon}\big| \leqslant C\left(|||f|||_p\sup_{s\in[0,1]}\pi_s\bar{V}^{(2[(2p)\vee p_0]+1/2)}\right)^2\Big\{1 + K^{-2}\|\nabla\phi\|_{p_0}h + \frac{\daleth(\alpha_{2p}\vee\alpha_p)}{K}\mu_0\bar{V}^{(2p+1/2)}\Big\}^2(\epsilon\vee h).$$

□

Lemma 53. *For any $\daleth>1$ and $\epsilon,h,K>0$ such that $\daleth^{-1}<1-K\epsilon^{-1}h/2$ we have,*

$$\big|\Upsilon_{4,\epsilon}\big| \leqslant \Upsilon^{(1)}_{4,\epsilon} + \Upsilon^{(2)}_{4,\epsilon}$$

where, with the convention $(\ell\vee 1)/\ell = 1$ for $\ell=0$, for any $\zeta\in(0,1)$, with

$$C_{fg} := (1+|||f|||_p)^2\tilde{\alpha}_p\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)}\left\{\frac{\daleth}{K} + \frac{C(\daleth,\zeta)(\ell\vee 1)}{(1\wedge K)\ell}\Big(2+\tilde{\alpha}_p\frac{M}{K}\sup_{\tau\in[0,1]}\sqrt{\bar{V}(x^\star_\tau)}\Big)\right\},$$

$$\Upsilon^{(1)}_{4,\epsilon} := Ch^\zeta\tilde{\alpha}_{2p+1/2}\big[C_{fg}\vee\big(\daleth\frac{\tilde{\alpha}_p}{K}|||f|||_p^2\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)}\big)\big]\left[1+\tilde{\alpha}_{2p+1/2}\frac{M}{K}\sup_{s\in[0,1]}\sqrt{\bar{V}(x^\star_s)}\right]$$
$$\Upsilon^{(2)}_{4,\epsilon} := C\daleth\|f\|_p^2\frac{\tilde{\alpha}_p}{K}\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)}\cdot\sup_{s\in[0,1]}\pi_s\big(\bar{V}^{(2p+1/2)}\big)\epsilon$$

and for $\ell > 0$

$$\left|\Upsilon_{6,\epsilon}\right| \leqslant C\ell h \|\!|f|\!\|_p^2 (\tilde{\alpha}_{2p} \vee \tilde{\alpha}_p) \sup_{s\in[0,1]} \pi_s \bar{V}^{(p)} \left[1 + (\tilde{\alpha}_{2p} \vee \tilde{\alpha}_p)\frac{M}{K} \sup_{s\in[0,1]} \sqrt{\bar{V}(x_s^\star)}\right]$$

while for $\ell = 0$ we have

$$\left|\Upsilon_{6,\epsilon}\right| \leqslant \sup_{s\in[0,1]} \mathrm{var}_{\pi_s}\left(f_s\right) \cdot h\epsilon^{-1}.$$

Proof. For $\Upsilon_{4,\epsilon}$, with the notation of Lemma 62, we introduce for $r \in \mathbb{N}$

$$\Upsilon^{(1)}_{4,\epsilon,r} := 2\left| h\Big\{\sum_{k=1}^{n-2} \pi_{kh}\left(\bar{f}_{kh} g_{kh,r}\right)\Big\} - \int_0^1 \pi_s(\bar{f}_s g_{s,r})\mathrm{d}s\right|,$$

$$\Upsilon^{(2)}_{4,\epsilon,r} := 2\left| h\Big\{\sum_{k=1}^{n-2} \pi_{kh}\left(\bar{f}_{kh} \Delta_{kh,r}\right)\Big\} - \int_0^1 \pi_s(\bar{f}_s \Delta_{kh,r})\mathrm{d}s\right|.$$

From the rough upper bound on $\Delta_{s,r}$ in Lemma 62 and with $r(0,\epsilon) := \lfloor -\ln(\epsilon)/K\rfloor$ or $r(\ell,\epsilon) := \lfloor -\ln(\epsilon)/(\ell K)\rfloor$ for $\ell > 0$, we have

$$\begin{aligned}\Upsilon^{(2)}_{4,\epsilon,r(\ell,\epsilon)} &\leqslant C\beth \|f\|_p \frac{\tilde{\alpha}_p}{K} \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)} \cdot \sup_{s\in[0,1]} \pi_s\big(\bar{f}_s \bar{V}^{(p+1/2)}\big)\epsilon \\ &\leqslant C\beth \|f\|_p^2 \frac{\tilde{\alpha}_p}{K} \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)} \cdot \sup_{s\in[0,1]} \pi_s\big(\bar{V}^{(2p+1/2)}\big)\epsilon.\end{aligned}$$

For the other terms we note that from Lemma 62 for $|s-t| \leqslant R_f = 1$ and any $\zeta \in (0,1)$

$$\begin{aligned}\left|f_s(x)g_{s,r}(x) - f_t(x)g_{t,r}(x)\right| &\leqslant \left|f_s(x) - f_t(x)\right|\cdot\left|g_{s,r}(x)\right| + \left|f_t(x)\right|\left|g_{s,r}(x) - g_{t,r}(x)\right| \\ &\leqslant C\frac{\beth\tilde{\alpha}_p}{K}\|f\|_p \bar{V}^{(p+1/2)}(x) \sup_{s\in[0,1]} \pi_s \bar{V}^{(p+1/2)}\left|f_s(x) - f_t(x)\right| + \|f\|_p \bar{V}^{(p)}(x)\left|g_{s,r}(x) - g_{t,r}(x)\right|\end{aligned}$$

Now, since $f \in C^p_{1,2}\big([0,1]\times\mathbb{R}^d\big)$,

$$\left|f_s(x) - f_t(x)\right| \leqslant \|\partial f\|_p \bar{V}^{(p)}(x)|s-t|$$

and from Lemma 62

$$\begin{aligned}|g_{s,r}(x) - g_{u,r}(x)| \leqslant C(\beth,\zeta)|s-u|^\zeta \frac{\tilde{\alpha}_p(\ell\vee 1)}{(1\wedge K)\ell}\big(1 \vee \|\!|f|\!\|_p\big) \\ \times\Big(1 + \tilde{\alpha}_p\frac{M}{K}\sup_{\tau\in[0,1]}\sqrt{\bar{V}(x_\tau^\star)} + \sup_{s\in[0,1]}\pi_s \bar{V}^{p+1/2}\Big).\end{aligned}$$

Hence, using Lemma 56

$$\begin{aligned}&\left|f_s(x)g_{s,r}(x) - f_t(x)g_{t,r}(x)\right| \\ &\quad\leqslant C(1+\|\!|f|\!\|_p)^2 \bar{V}^{(2p+1/2)}(x)\tilde{\alpha}_p \sup_{s\in[0,1]}\pi_s \bar{V}^{(p+1/2)}\left\{\frac{\beth}{K} + \frac{C(\beth,\zeta)(\ell\vee 1)}{(1\wedge K)\ell}\Big(2 + \tilde{\alpha}_p\frac{M}{K}\sup_{\tau\in[0,1]}\sqrt{\bar{V}(x_\tau^\star)}\Big)\right\}|s-t|^\zeta\end{aligned}$$

where $C(\beth,\xi)$ depends on the arguments shown only. Now, defining

$$C_{fg} := (1+\|\!|f|\!\|_p)^2\tilde{\alpha}_p \sup_{s\in[0,1]}\pi_s \bar{V}^{(p+1/2)}\left\{\frac{\beth}{K} + \frac{C(\beth,\zeta)(\ell\vee 1)}{(1\wedge K)\ell}\Big(2 + \tilde{\alpha}_p\frac{M}{K}\sup_{\tau\in[0,1]}\sqrt{\bar{V}(x_\tau^\star)}\Big)\right\}$$

from Lemma 63

$$\Upsilon^{(1)}_{4,\epsilon,r} \leqslant Ch^\zeta\tilde{\alpha}_{2p+1/2}\big(C_{fg} \vee \|\nabla fg\|_{2p+1/2}\big)\left[1 + \tilde{\alpha}_{2p+1/2}\frac{M}{K}\sup_{s\in[0,1]}\sqrt{\bar{V}(x_s^\star)}\right]$$

and we have found in the proof of Lemma 52 that

$$\sup_{r\in\mathbb{N}}\|\nabla(fg_r)\|_{2p+1/2}\leqslant C\beth\frac{\tilde{\alpha}_p}{K}\|\!|f|\!\|_p^2\sup_{s\in[0,1]}\pi_s\bar{V}^{(p+1/2)},$$

from which the first bound follows. For $\Upsilon_{6,\epsilon}$, first consider $\ell=0$. In this case

$$\left|\Upsilon_{6,\epsilon}\right|\leqslant h\epsilon^{-1}\sup_{s\in[0,1]}\operatorname{var}_{\pi_s}\big(f_s\big)$$

Now consider $\ell>0$, we apply Lemma 63 with the function f and f^2 to obtain the resul. By assumptions $f\in C^p_{1,2}\big([0,1]\times\mathbb{R}^d\big)$ implies that

$$\big|f_s(x)-f_t(x)\big|\leqslant\|\!|f|\!\|_p\bar{V}_p(x)|s-t|$$

and consequently

$$\left|h\sum_{k=0}^{\lfloor 1/h\rfloor-1}\big(\pi_{kh}f_{kh}\big)^2-\int_0^1\big(\pi_tf_t\big)^2\mathrm{d}t\right|\leqslant h\tilde{\alpha}_p\|\!|f|\!\|_p\left[1+\tilde{\alpha}_p\frac{M}{K}\sup_{s\in[0,1]}\sqrt{\bar{V}(x_s^\star)}\right]\cdot 2\|f\|_p\sup_{s\in[0,1]}\pi_s\bar{V}^{(p)}$$

Further,

$$\begin{aligned}\big|f_s^2(x)-f_t^2(x)\big|&\leqslant 2\|\!|f|\!\|_p^2\big[\bar{V}^{(p)}(x)\big]^2|s-t|\\&\leqslant C\|\!|f|\!\|_p^2\bar{V}^{(2p)}(x)|s-t|\end{aligned}$$

and by application of Lemma 63 we deduce

$$\big|\Upsilon_{6,\epsilon}\big|\leqslant C\ell h\|\!|f|\!\|_p^2(\tilde{\alpha}_{2p}\vee\tilde{\alpha}_p)\sup_{s\in[0,1]}\pi_s\bar{V}^{(p)}\left[1+(\tilde{\alpha}_{2p}\vee\tilde{\alpha}_p)\frac{M}{K}\sup_{s\in[0,1]}\sqrt{\bar{V}(x_s^\star)}\right].$$

□

Lemma 54. *There exists $C>0$ such that for any $\beth>1$ and $\epsilon,h>0$ and $K>0$ satisfying $\beth^{-1}<1-Kh\epsilon^{-1}/2$ $\epsilon>0$ we have*

$$\begin{aligned}\big|\Upsilon_{0,\epsilon}\big|&\leqslant C\alpha_p\alpha_{p+1/2}\frac{\beth\|\nabla f\|_p^2}{K}\cdot\Big\{\mu_0\bar{V}^{(p+1/2)}(x)\Big\}^2\sup_{s\in[0,1]}\pi_s\bar{V}^{(2[p\vee p_0]+1/2)}\\&\qquad\times\Big\{K^{-2}\|\nabla\phi\|_{p_0}+\alpha_p\mu_0\bar{V}^{(p+1/2)}\frac{\exp\big(-K\epsilon^{-1}h\big)}{1-\exp\big(-K\epsilon^{-1}h\big)}\epsilon^{-1}h\Big\}\epsilon\\\big|\Upsilon_{7,\epsilon}\big|&\leqslant C\epsilon^{-1}h^2\|f\|^2_{V^{(p)}}\alpha_{2p}\mu_0V^{(2p)}+C\frac{\beth}{K}\|\nabla f\|\big\{\alpha_{2p}\big[K^{-1}+K_{\mu_0}\big]\mu_0\bar{V}^{(2p)}\big\}^{1/2}h.\end{aligned}$$

Proof. First we have the simplification

$$\begin{aligned}\Upsilon_{0,\epsilon}&=\epsilon^{-1}h^2\sum_{k=1}^{n-2}\pi_{kh}f_{kh}\mathbb{E}\big[2\gamma_{k,\epsilon}(X_{kh}^\epsilon)-f_{kh}(X_{kh}^\epsilon)\big]\\&=2\epsilon^{-1}h^2\sum_{k=1}^{n-2}\pi_{kh}f_{kh}\mathbb{E}\big[\gamma_{k,\epsilon}(X_{kh}^\epsilon)\big],\end{aligned}$$

and from Lemma 37,

$$\begin{aligned}\big|\mathbb{E}\left(\gamma_{k,\epsilon}(X_{kh}^\epsilon)\right)\big|&\leqslant C\alpha_p\frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)}\cdot\sup_{s\in[0,1]}\mu_s\bar{V}^{(p+1/2)}(x)\\&\leqslant C\alpha_p\alpha_{p+1/2}\frac{\|\nabla f\|_p}{1-\exp\big(-K\epsilon^{-1}h\big)}\mu_0\bar{V}^{(p+1/2)}\cdot\mu_0\bar{V}^{(p+1/2)}\end{aligned}$$

where we have used Lemma 14 in the last line. Further from Lemma 58

$$\big|\pi_t f_t\big| = |\mathbb{E}\big[\bar{f}_t(X_t^\epsilon)\big]| \leqslant C\|\nabla f\|_p \cdot \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p\vee p_0]+1/2)} \Big\{K^{-2}\|\nabla\phi\|_{p_0}\epsilon + \alpha_p\mu_0\bar{V}^{(p+1/2)}\exp\big(-K\epsilon^{-1}t\big)\Big\}$$

and therefore

$$2\epsilon^{-1}h^2\sum_{k=1}^{n-2}\big|\pi_{kh}f_{kh}\big| \leqslant C\|\nabla f\|_p \cdot \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p\vee p_0]+1/2)} \Big\{K^{-2}\|\nabla\phi\|_{p_0}h + \alpha_p\mu_0\bar{V}^{(p+1/2)}\frac{\exp\big(-K\epsilon^{-1}h\big)}{1-\exp\big(-K\epsilon^{-1}h\big)}\epsilon^{-1}h^2\Big\}$$

We have

$$\Upsilon_{7,\epsilon} := \epsilon^{-1}h^2\mathbb{E}\Big[f^2_{(n-1)h}(X^\epsilon_{(n-1)h}) - \big[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\big]^2\Big].$$

We have $\mathbb{E}\Big[f^2_{(n-1)h}(X^\epsilon_{(n-1)h})\Big] \leqslant C\|f\|_p^2\alpha_{2p}\mu_0 V^{(2p)}$. Now we notice that

$$\begin{aligned}
\epsilon^{-1}h\mathbb{E}\Big[\big[P_{0,h}\gamma_{1,\epsilon}(X_0^\epsilon)\big]^2\Big]^{1/2} &= \epsilon^{-1}h\mathbb{E}\Bigg[\Big(\sum_{i=1}^{n-1}P_{0,ih}f_{ih}(X_0^\epsilon)\Big)^2\Bigg]^{1/2} \\
&\leqslant \epsilon^{-1}h\sum_{i=1}^{n-1}\mathrm{var}_{\mu_0}\big(P_{0,ih}f_{ih}\big)^{1/2} \\
&\leqslant \frac{\epsilon^{-1}h}{1-\exp(-Kh\epsilon^{-1})}\sup_{s\in[0,1]}\mathrm{var}_{\mu_s}\big(f_s\big)^{1/2} \\
&\leqslant C\frac{\gimel}{K}\|\nabla f\|\big\{\alpha_{2p}\big[K^{-1}+K_{\mu_0}\big]\mu_0\bar{V}^{(2p)}\big\}^{1/2}.
\end{aligned}$$

□

In Proposition 43 it is required to control the L_q convergence of the term D_ϵ defined above Proposition 42, which is an ergodic average. It is possible to get estimates of this quantity by using a Martingale approximation, followed by the use of Burkholder's inequality. We however use here a more direct route since no precise estimates are needed.

Lemma 55. *Let $p \geqslant 1$, $f \in C^p_{0,2}([0,1]\times\mathbb{R}^d)$ and $q \geqslant 1$. Then for any $r > 1 \vee (2/q)$ and with $m = (qr-2)/(r-1)$*

$$\|S_{\epsilon,h} - \mathbb{E}\big[S_{\epsilon,h}\big]\|_{L_q} \leqslant C\left(\|S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\|_{L_2}\right)^{\frac{2}{qr}}\|f\|_p^{1-\frac{2}{qr}}\left(\alpha_{pm}^{1/m}\left(\mu_0\bar{V}^{(pm)}\right)^{1/m} + \alpha_p\mu_0\bar{V}^{(p)}\right)^{1-\frac{2}{qr}}.$$

Proof. Let $l := m/(q-\frac{2}{r})$, then $r^{-1}+l^{-1} = 1$ and we apply Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left[\big(S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\big)^q\right] &= \mathbb{E}\left[\big(S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\big)^{\frac{2}{r}}\big(S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\big)^{q-\frac{2}{r}}\right] \\
&\leqslant \mathbb{E}\left[\big(S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\big)^2\right]^{1/r}\mathbb{E}\left[\big(S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\big)^{(q-\frac{2}{r})l}\right]^{1/l}.
\end{aligned}$$

Using the triangle inequality we get

$$\|S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\|_{L_q} \leqslant \big(\|S_{\epsilon,h}-\mathbb{E}\big[S_{\epsilon,h}\big]\|_{L_2}\big)^{\frac{2}{qr}}\left(\|S_{\epsilon,h}\|_{L_m} + \|f\|_p\sup_{t\in[0,1]}\mu_t\bar{V}^{(p)}\right)^{1-\frac{2}{qr}}.$$

Now, noting that $\mathbb{E}\big[S_{\epsilon,h}\big] = h\sum_{i=0}^{n-1}\mu_{ih}f_{ih}$, by the triangle inequality and from Lemma 56 and Lemma 14

$$\begin{aligned}
\|S_{\epsilon,h}\|_{L_m} &\leqslant h\sum_{i=0}^{n-1}\|f_{ih}\|_p\mathbb{E}\left[\bar{V}^{(p)}\big(X_{ih}\big)^m\right]^{1/m} \\
&\leqslant \|f\|_p 2^{m-1}h\sum_{i=0}^{n-1}\mathbb{E}\left[\bar{V}^{(pm)}\big(X_{ih}\big)^m\right]^{1/m}. \\
&\leqslant \|f\|_p 2^{m-1}\alpha_{pm}^{1/m}\left(\mu_0\bar{V}^{(pm)}\right)^{1/m}.
\end{aligned}$$

$\square$

5.5 Rough, but tractable, bounds

We gather here intermediate technical results which lead to tractable bounds and allow us to conclude about the complexity of the procedure. For the reader's convenience we recall that for $q > 0$ and $x \in \mathbb{R}^d$, $V(x) := \|x\|^2$, $V^{(q)} := V^q$, $\bar{V}^{(q)} := 1 + V^{(q)}$, with $t \in [0,1]$ $V_t(x) := \|x - x_t^\star\|^2$, $V_t^{(q)} := V_t^q$, $\bar{V}_t^{(q)} := 1 + V_t^{(q)}$ (with notational simplifications $\bar{V}_t := \bar{V}_t^{(1)}$ and $V_t := V_t^{(1)}$ etc.) and for $\nu \in \mathcal{P}^{q+1/2}(\mathbb{R}^d)$

$$W^{(q)}(\delta_x, \nu) := \int_{\mathbb{R}^d} \left(1 + \|x\|^{2q} \vee \|y\|^{2q}\right) \|x - y\| \nu(\mathrm{d}y).$$

Lemma 56. *For any $p \geqslant 1$ and $\nu \in \mathcal{P}^{p+1/2}(\mathbb{R}^d)$,*

$$W^{(p)}(\delta_x, \nu) \leqslant V^{p+1/2}(x) + V^p(x)\nu(V^{1/2}) + V^{1/2}(x)[1 + \nu(V^p)] + \nu(V^{p+1/2}), \quad x \in \mathbb{R}^d,$$

and as a result

$$\sup_{x \in \mathbb{R}^d} \frac{W^{(p)}(\delta_x, \nu)}{1 + \|x\|^{2p+1}} < +\infty.$$

Further there exists $C > 0$ such that for any $x \in \mathbb{R}^d$ and $\nu \in \mathcal{P}^{p+1/2}(\mathbb{R}^d)$

$$W^{(p)}(\delta_x, \nu) \leqslant C\nu\bar{V}^{(p+1/2)} \cdot \bar{V}^{(p+1/2)}(x). \tag{98}$$

Proof. By considering the scenarios $\|x\| \leqslant \|y\|$ and $\|x\| > \|y\|$ separately we have

$$\begin{aligned} W^{(p)}(\delta_x, \nu) &\leqslant \|x\| + \nu(V^{1/2}) + \|x\|^{2p+1} + \|x\|^{2p}\nu(V^{1/2}) + \|x\|\nu(V^p) + \nu(V^{p+1/2}), \\ &= \|x\|^{2p+1} + \|x\|^{2p}\nu(V^{1/2}) + \|x\|[1 + \nu(V^p)] + \nu(V^{p+1/2}), \end{aligned}$$

and the first statement follows from the assumption on ν. Finally by considering the scenarios $V(x) \geqslant 1$ and $V(x) < 1$ separately twice one shows that

$$\begin{aligned} W^{(p)}(\delta_x, \nu) &\leqslant 2\left[1 + V^{p+1/2}(x)\right]\left[1 + \nu\left(V^{1/2} + V^p + V^{p+1/2}\right)\right], \\ &\leqslant 8\nu\bar{V}^{(p+1/2)} \cdot \bar{V}^{(p+1/2)}(x). \end{aligned}$$

$\square$

Lemma 57. *For any $p \geqslant 0$,*

1. *for any $q \geqslant 0$ and $x \in \mathbb{R}^d$*

$$\bar{V}^{(p)}(x)\bar{V}^{(q)}(x) \leqslant 4 \cdot \bar{V}^{(p+q)}(x),$$

$$V^{(p)}(x) \vee V^{(q)}(x) \leqslant 2 \cdot V^{(p \vee q)}(x),$$

for any $q \geqslant 1$

$$\left[\bar{V}^{(p)}(x)\right]^q \leqslant 2^{q-1}\bar{V}^{(qp)}(x),$$

and for $\varphi, \psi \in C^p\left(\mathbb{R}^d\right) \times C^q\left(\mathbb{R}^d\right)$ for $p, q \geqslant 1$

$$\|\varphi\psi\|_{p+q} \leqslant 4\|\varphi\|_p\|\psi\|_q$$

2. *for any $s \in [0,1]$ and $x \in \mathbb{R}^d$,*

$$\sqrt{\bar{V}_s(x)}\bar{V}^{(p)}(x) \leqslant \sqrt{12}\bar{V}(x_s^\star)^{1/2}\bar{V}^{(p+1/2)}(x)$$

Proof. First we have $\bar{V}^{(p)}(x)\bar{V}^{(q)}(x) \leqslant 4\bar{V}^{(p+q)}(x)$ because $\bar{V}^{(p)}(x)\bar{V}^{(q)}(x) = 1 + \|x\|^{2(p+q)} + \|x\|^{2q} + \|x\|^{2p}$ and one can consider the scenarios $\|x\| \geqslant 1$ and $\|x\| < 1$ separately. For the second statement one can again consider the scenarios $\|x\| \geqslant 1$ and $\|x\| < 1$. For the third statement, the result follows from Jensen's inequality,

$$\big[1 + \|x\|^{2p}\big]^q \leqslant 2^q \frac{1 + \|x\|^{2pq}}{2}.$$

The next statement follows from

$$\frac{\varphi(x)\psi(x)}{\bar{V}^{(p+q)}(x)} = \frac{\varphi(x)\psi(x)}{\bar{V}^{(p)}(x)\bar{V}^{(q)}(x)} \frac{\bar{V}^{(p)}(x)\bar{V}^{(q)}(x)}{\bar{V}^{(p+q)}(x)}$$

and our first result above. Now we note that for $z \geqslant 0$ and $C > 0$

$$\begin{aligned} A(z) &:= (C + z)(1 + z^p)^2 = z^{2p+1} + Cz^{2p} + 2[z^{p+1} + Cz^p] + z + C \\ B(z) &:= (1 + z^{p+1/2})^2 = z^{2p+1} + 2z^{p+1/2} + 1 \end{aligned}$$

are such that for $z \geqslant 1$ $A(z) \leqslant z^{2p+1}[1+C+2(1+C)+1+C]$ and for $z \leqslant 1$ $A(z) \leqslant [1+C+2(1+C)+1+C]$, and therefore for $z \geqslant 0$

$$\begin{aligned} A(z) &\leqslant 4(1 + z^{2p+1})[1 + C] \\ &\leqslant 4(1 + C)B(z) \end{aligned}$$

as a consequence with $C = 1/2 + \|x_s^\star\|^2$ and $z = \|x\|^2$ we deduce that (with $\|x - x_s^\star\|^2 \leqslant 2[\|x\|^2 + \|x_t^\star\|^2)]$)

$$\sqrt{2}\sqrt{1/2 + 1/2\|x - x_s^\star\|^2}(1 + \|x\|^{2p}) \leqslant \sqrt{8(1 + 1/2 + \|x_s^\star\|^2)}\big(1 + \|x\|^{2p+1}\big)$$

that is

$$\sqrt{\bar{V}_s(x)} \cdot \bar{V}^{(p+1/2)}(x) \leqslant \sqrt{12\bar{V}(x_s^\star)} \cdot \bar{V}^{(p+1/2)}(x)$$

$\square$

Lemma 58.

1. *There exists $C > 0$ such that for any $p \geqslant 1$, $\nu \in \mathcal{P}^{2p}(\mathbb{R}^d)$ such that there exists a constant $K_\nu > 0$ such that for all $f \in C_2^p(\mathbb{R}^d)$*

$$\mathrm{var}_\nu[f] \leqslant K_\nu^{-1}\nu\big(\|\nabla f\|^2\big),$$

then for any $f \in C_2^p(\mathbb{R}^d)$ and $\epsilon > 0$

$$\begin{aligned} \sup_{0 \leqslant s \leqslant t \leqslant 1} \mathrm{var}_{\nu P_{s,t}^\epsilon}\big[f\big] &\leqslant C\alpha_{2p} \cdot \|\nabla f\|_p^2 \cdot \big[K^{-1} + K_\nu^{-1}\big]\, \nu(\bar{V}^{(2p)}) \\ \sup_{(s,t) \in [0,1] \times \mathbb{R}_+} \mathrm{var}_{\nu Q_t^{s,\epsilon}}\big[f\big] &\leqslant C\tilde{\alpha}_{2p} \cdot \|\nabla f\|_p^2 \cdot \big[K^{-1} + K_\nu^{-1}\big]\, \nu(\bar{V}^{(2p)}) \end{aligned}$$

where α_{2p} and $\tilde{\alpha}_{2p}$ are given in Lemma 14 and 61 respectively.

2. *There exists $C > 0$ such that for any ϕ_t as in (11),*

$$\sup_{t \in [0,1]} \mathrm{var}_{\pi_t}[\phi_t] \leqslant CK^{-1}\|\nabla \phi\|_{p_0}^2 \cdot \sup_{t \in [0,1]} \pi_t\big(\bar{V}^{(2p_0)}\big).$$

3. *Let $p \geqslant 1$, then for any $f \in C_2^p(\mathbb{R}^d)$*

$$\begin{aligned} |\mathbb{E}\big[\bar{f}_t(X_t^\epsilon)\big]| \leqslant \sup_{s \in [0,1]} \mathrm{var}_{\pi_s}[\phi_s]^{1/2} \sup_{s \in [0,1]} \mathrm{var}_{\pi_s}[f_s]^{1/2} \frac{\epsilon}{K} \big[1 - \exp(-K\epsilon^{-1}t)\big] \\ + \alpha_p \|\nabla f_t\|_p W^{(p)}(\mu_0, \pi_0) \exp\big(-K\epsilon^{-1}t\big) \end{aligned}$$

and a rough bound is

$$|\mathbb{E}\big[\bar{f}_t(X_t^\epsilon)\big]| \leqslant C\|\nabla f\|_p \cdot \sup_{s \in [0,1]} \pi_s \bar{V}^{(2[p \vee p_0]+1/2)} \Big\{K^{-2}\|\nabla \phi\|_{p_0}\epsilon + \alpha_p \mu_0 \bar{V}^{(p+1/2)} \exp\big(-K\epsilon^{-1}t\big)\Big\}.$$

Corollary 59. *As a consequence for $t \in [0,1]$*

$$\mathrm{var}_{\mu_t^\epsilon}\big[f\big] \leqslant C\alpha_{2p} \cdot \|\nabla f\|_p^2 \left[K^{-1} + K_{\mu_0}^{-1}\right] \mu_0(\bar{V}^{(2p)})$$

and using Lemmas 22 and 14 for any $(s,t) \in [0,1] \times \mathbb{R}_+$

$$\mathrm{var}_{\mu_s Q_t^{s,\epsilon}}\big[f\big] \leqslant C\tilde{\alpha}_{2p}\alpha_{2p} \cdot \|\nabla f\|_p^2 \left[K^{-1} + K_{\mu_0}^{-1}\right] \mu_s(\bar{V}^{(2p)})$$

and

Proof. We first apply Lemma 22, yielding for $0 \leqslant s \leqslant t \leqslant 1$

$$\begin{aligned}\mathrm{var}_{\nu P_{s,t}^\epsilon}\big[f\big] &\leqslant \left[K^{-1} + K_\nu^{-1}\right] \cdot \nu P_{s,t}\left(\|\nabla f\|^2\right) \\ &\leqslant \left[K^{-1} + K_\nu^{-1}\right] \|\nabla f\|_p^2 \cdot \nu P_{s,t}\left([\bar{V}^{(p)}]^2\right).\end{aligned}$$

Now we apply (46) in Lemma 14 and Lemma 57 to conclude. We proceed similarly for the time homogeneous scenario and Lemma 61. We use Remark 21 noting the fact, established in the proof of Lemma 25, that $\phi_t \in C_{0,2}^{p_0}([0,1] \times \mathbb{R}^d)$. As a result for $t \in [0,1]$ we have

$$\begin{aligned}\mathrm{var}_{\pi_t}[\phi_t] &\leqslant K^{-1}\pi_t\left(\|\nabla \phi_t\|^2\right) \\ &\leqslant K^{-1}\|\nabla \phi_t\|_{p_0}^2 \pi_t\left([\bar{V}^{(p_0)}]^2\right),\end{aligned}$$

and we conclude with Lemma 57. For the bias, we note that for $t \in [0,1]$

$$\mathbb{E}[f_t(X_t)] = \mu_0 P_{0,t} f_t = \pi_0 P_{0,t} f_t - \pi_t f_t + (\mu_0 - \pi_0) P_{0,t} f_t,$$

and by Lemmas 24 and 26, we deduce

$$|\mathbb{E}\big[\bar{f}_t(X_t^\epsilon)\big]| \leqslant \sup_{s\in[0,1]} \mathrm{var}_{\pi_s}[\phi_s]^{1/2} \sup_{s\in[0,1]} \mathrm{var}_{\pi_s}[f_s]^{1/2} \frac{\epsilon}{K}\left[1 - \exp(-K\epsilon^{-1}t)\right] + \alpha_p \|\nabla f_t\|_p W^{(p)}(\mu_0, \pi_0) \exp\left(-K\epsilon^{-1}t\right).$$

We can now apply our earlier result and Remark 21 to show,

$$\begin{aligned}&\sup_{s\in[0,1]} \mathrm{var}_{\pi_s}[\phi_s]^{1/2} \sup_{s\in[0,1]} \mathrm{var}_{\pi_s}[f_s]^{1/2} \frac{\epsilon}{K}\left[1 - \exp(-K\epsilon^{-1}t)\right] \\ &\qquad \leqslant CK^{-2}\|\nabla\phi\|_{p_0}\|\nabla f\|_p \cdot \left\{\sup_{s\in[0,1]} \pi_s \bar{V}^{(2p_0)} \cdot \sup_{s\in[0,1]} \pi_s \bar{V}^{(2p)}\right\}^{1/2} \epsilon \\ &\qquad \leqslant CK^{-2}\|\nabla\phi\|_{p_0}\|\nabla f\|_p \cdot \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p_0 \vee p])}\epsilon\end{aligned}$$

and from Lemma 56

$$\alpha_p \|\nabla f_t\|_p W^{(p)}(\mu_0, \pi_0) \exp\left(-K\epsilon^{-1}t\right) \leqslant C\alpha_p \|\nabla f\|_p \mu_0 \bar{V}^{(p+1/2)} \cdot \pi_0 \bar{V}^{(p+1/2)} \exp\left(-K\epsilon^{-1}t\right)$$

from which we deduce

$$|\mathbb{E}\big[\bar{f}_t(X_t^\epsilon)\big]| \leqslant C\|\nabla f\|_p \sup_{s\in[0,1]} \pi_s \bar{V}^{(2[p\vee p_0]+1/2)} \left\{K^{-2}\|\nabla\phi\|_{p_0}\epsilon + \alpha_p \mu_0 \bar{V}^{(p+1/2)} \exp\left(-K\epsilon^{-1}t\right)\right\}.$$

□

Lemma 60. *For $0 \leqslant z < 2$*

$$\frac{z}{1 - \exp(-z)} \leqslant \frac{1}{1 - z/2}$$

Proof. We have that for $z \geqslant 0$ $\exp(-z) \leqslant 1 - z + \frac{z^2}{2}$, which implies $[1 - \exp(-z)]/z \geqslant 1 - z/2$ and therefore the result. □

6 Drift and solution of Poisson's equation for the time-homogeneous diffusions

Throughout section 6 the notational conventions of section 5 are in force, except that f_t is not assumed centred with respect to π_t, and we write $\bar{f}_t :== f_t - \pi_t f_t$.

Lemma 61. *For any $\epsilon > 0$, $p \geqslant 1$ and $\kappa \in (0, Kp)$, define*

$$\begin{aligned}
\delta &:= \epsilon^{-1}(Kp - \kappa), \\
\tilde{r} &:= \sqrt{\frac{4p(p-1) + 2pd}{\kappa}} \\
\tilde{b} &:= 2p\tilde{r}^{2(p-1)} \frac{2(p-1) + d}{\epsilon} \\
\tilde{\alpha}_p &:= 2^{4p-2} \vee \left[1 + 2^{2p-1} \left(\frac{2p\tilde{r}^{2(p-1)}}{(Kp - \kappa)} [2(p-1) + d] + (1 + 2^{2p-1}) \sup_{t \in [0,1]} \|x_t^\star\|^{2p} \right) \right]
\end{aligned}$$

Then

$$\begin{aligned}
Q_t^{s,\epsilon}(V_s^p)(x) &\leqslant e^{-\delta t} V_s^p(x) + \frac{\tilde{b}}{\delta}(1 - e^{-\delta t}), \quad \forall (s,t) \in [0,1] \times \mathbb{R}_+, \qquad (99) \\
\sup_{(s,t) \in [0,1] \times \mathbb{R}_+} Q_t^{s,\epsilon} \bar{V}^{(p)}(x) &\leqslant \tilde{\alpha}_p \bar{V}^{(p)}(x).
\end{aligned}$$

Proof. The result follows by almost identical arguments to those in the proof of Lemma 14, with some elementary simplifications afforded by the time-homogeneity of the process $Y_t^{s,\epsilon}$. □

Lemma 62. *Let $p \geqslant 1$ and $f \in C_{0,2}^p([0,1] \times \mathbb{R}^d)$ such that for constants $C_f < +\infty$, $R_f \in (0,1]$ and $\beta \in (0,1]$*

$$|s - u| \leqslant R_f \quad \Rightarrow \quad |f_s(x) - f_u(x)| \leqslant C_f |s - u|^\beta \bar{V}^{(p)}(x), \quad \forall x \in \mathbb{R}^d, \qquad (100)$$

and define for any $s \in [0,1]$ and $r \in \mathbb{N} \cup \{\infty\}$,

$$g_{s,r}(x) := \begin{cases} \sum_{k=0}^r \ell Q_{k\ell}^s \bar{f}_s(x), & \text{if} \quad \ell > 0, \\ \int_0^r Q_t^s \bar{f}_s(x) \mathrm{d}t, & \text{if} \quad \ell = 0. \end{cases}$$

Then, with $\tilde{\alpha}_{p,1}$ as in Lemma 61 with there $\epsilon = 1$,

1. *we have*

$$|s - u| \leqslant R_f \quad \Rightarrow \quad |\pi_s f_s - \pi_u f_u| \leqslant C|s - u|^\beta \tilde{\alpha}_p (C_f \vee \|\nabla f\|_p) \left[1 + \tilde{\alpha}_{p,1} \frac{M}{K} \sup_{\tau \in [0,1]} \sqrt{\bar{V}(x_\tau^\star)} \right] \quad \forall x \in \mathbb{R}^d. \qquad (101)$$

2. *$g_{s,r}(\cdot)$ has the following properties:*

   (a) *for any $\ell \geqslant 0$, $s \in [0,1]$ and $r < \infty$, the map $x \mapsto g_{s,r}(x)$ is a member of $C_2^p(\mathbb{R}^d)$,*

   (b) *for any $\ell \geqslant 0$, $s \in [0,1]$ and $r \in \mathbb{N} \cup \{\infty\}$,*

$$|g_{s,r}(x)| \leqslant \begin{cases} h\epsilon^{-1} \|f\|_p \tilde{\alpha}_p W^{(p)}(\delta_x, \pi_s) \frac{1}{1 - e^{-Kh\epsilon^{-1}}}, & \ell > 0, \\ \|f\|_p \tilde{\alpha}_p W^{(p)}(\delta_x, \pi_s) \frac{1}{K}, & \ell = 0 \end{cases}$$

   and further for any $\daleth > 1$ and $\daleth^{-1} \leqslant 1 - Kh\epsilon^{-1}/2$ we have the simplified upper bound

$$\sup_{(r,s) \in \mathbb{N} \cup \{\infty\} \times [0,1]} \|g_{s,r}\|_{p+1/2} \leqslant C\daleth \frac{\tilde{\alpha}_{p,1}}{K} \|f\|_p \sup_{s \in [0,1]} \pi_s \bar{V}^{(p+1/2)}.$$

(c) for any $s \in [0,1]$, $r \in \mathbb{N} \cup \{\infty\}$ and $x \in \mathbb{R}^d$,

$$\Delta_{s,r}(x) := |g_{s,\infty}(x) - g_{s,r}(x)| \leqslant \begin{cases} \ell \|f\|_p \tilde{\alpha}_p W^{(p)}(\delta_x, \pi_s) \frac{e^{-Kh\epsilon^{-1}r}}{1-e^{-Kh\epsilon^{-1}}}, & \ell > 0, \\ \|f\|_p \tilde{\alpha}_p W^{(p)}(\delta_x, \pi_s) \frac{e^{-Kr}}{K}, & \ell = 0. \end{cases}$$

and further for any $\beth > 1$ and $\beth^{-1} \leqslant 1 - Kh\epsilon^{-1}/2$ we have the simplified upper bound

$$\sup_{(r,s) \in \mathbb{N} \cup \{\infty\} \times [0,1]} \|\Delta_{s,r}\|_{p+1/2} \leqslant C\beth \frac{\tilde{\alpha}_{p,1}}{K} \|f\|_p \sup_{s \in [0,1]} \pi_s \bar{V}^{(p+1/2)} \begin{cases} e^{-Kh\epsilon^{-1}r}, & \ell > 0, \\ e^{-Kr}, & \ell = 0. \end{cases}$$

(d) for any $\zeta \in (0, \beta)$ there exists $C > 0$ such that for any $\beth > 1$, $\beth^{-1} \leqslant 1 - K\ell/2$ if $\ell > 0$, $r \in \mathbb{N} \cup \{\infty\}$ and $x \in \mathbb{R}^d$,$|s-u| \leqslant R_f$

$$|g_{s,r}(x) - g_{u,r}(x)| \leqslant C(\beta, \beth, R_f, \zeta)|s-u|^\zeta \frac{\tilde{\alpha}_{p,1}(\ell \vee 1)}{(1 \wedge K)\ell} (C_f \vee \|f\|_p)$$
$$\times \left(1 + \tilde{\alpha}_{p,1} \frac{M}{K} \sup_{\tau \in [0,1]} \sqrt{\bar{V}(x_\tau^\star)} + \sup_{s \in [0,1]} \pi_s \bar{V}^{p+1/2}\right).$$

where $C(\beta, \beth, R_f, \zeta)$ depends only on the arguments shown and the convention that $(\ell \vee 1)/\ell = 1$ for $\ell = 0$.

Proof. Consider for arbitrary $s, u \in [0,1]$, $x \in \mathbb{R}^d$, and $t > 0$, the decomposition $\pi_s f_s - \pi_u f_u = R_1(t,x) + R_2(t,x) + R_3(t,x)$, where

$$R_1(t,x) := \pi_s f_s - Q_t^s f_s(x) + Q_t^u f_u(x) - \pi_u f_u,$$
$$R_2(t,x) := Q_t^s (f_s - f_u)(x),$$
$$R_3(t,x) := (Q_t^s - Q_t^u)(f_u)(x).$$

For R_1, it can be shown by arguments which are almost identical to those used to prove Lemma 24 that

$$|Q_t^s f_s(x) - \pi_s f_s| \leqslant \|f_s\|_p \tilde{\alpha}_{p,1} e^{-Kt} W^{(p)}(\delta_x, \pi_s). \tag{102}$$

Hence

$$|R_1(t,x)| \leqslant \|f\|_p \tilde{\alpha}_{p,1} e^{-Kt} \left[W^{(p)}(\delta_x, \pi_s) + W^{(p)}(\delta_x, \pi_u)\right],$$
$$\leqslant C\|f\|_p \tilde{\alpha}_{p,1} \sup_{s \in [0,1]} \pi_s \bar{V}^{(p+1/2)} \bar{V}^{(p+1/2)}(x) e^{-Kt},$$

where we have used the estimates of Lemma 56. For R_2, using (100) and Lemma 61,

$$\sup_{t \in \in \mathbb{R}_+} |R_2(t,x)| \leqslant C_f |s-u|^\beta \sup_{t \in \mathbb{R}_+} Q_t^s \bar{V}^{(p)}(x)$$
$$\leqslant C_f \tilde{\alpha}_{p,1} |s-u|^\beta \bar{V}^{(p)}(x).$$

For R_3, assuming w.l.o.g. that $u \leqslant s$,

$$\begin{aligned} |Q_t^s f_u - Q_t^u f_u| &= \left| \int_0^t \partial_\tau Q_\tau^u Q_{t-\tau}^s f_u \mathrm{d}\tau \right| \\ &= \left| \int_0^t Q_\tau^u \left\langle \nabla U_s - \nabla U_u, \nabla Q_{t-\tau}^s f_u \right\rangle \mathrm{d}\tau \right| \\ &\leqslant \int_0^t Q_\tau^u (\|\nabla U_s - \nabla U_u\| \|\nabla Q_{t-\tau}^s f_u\|) \mathrm{d}\tau \\ &\leqslant M|s-u| \int_0^t Q_\tau^u \left(\sqrt{\bar{V}_u} \cdot Q_{t-\tau}^s \|\nabla f_u\|\right) e^{-K(t-\tau)} \mathrm{d}\tau \\ &\leqslant \|\nabla f_u\|_p \tilde{\alpha}_{p,1} M|s-u| \int_0^t Q_\tau^u \left(\sqrt{\bar{V}_u} \cdot \bar{V}^{(p)}\right) e^{-K(t-\tau)} \mathrm{d}\tau. \end{aligned} \tag{103}$$

We now use Lemma 56 and Lemma 61,

$$\sup_{\tau\in[0,1]} Q^u_\tau\left(\sqrt{\bar V_u}\cdot \bar V^{(p)}\right)(x) \leqslant C\tilde\alpha_{p,1}\sqrt{\bar V(x^\star_u)}\cdot \bar V^{(p+1/2)}(x)$$

and combining this observation with (103) gives

$$\sup_{t\in\mathbb{R}_+} |R_3(t,x)| \leqslant C\tilde\alpha^2_{p,1}\frac{M}{K}|s-u|\cdot \|\nabla f\|_p\sqrt{\bar V(x^\star_u)}\cdot \bar V^{(p+1/2)}(x).$$

Since x was arbitrary we may now choose $x=0$, and noting also that t was arbitrary and $|s-u|\leqslant 1$, combining the above bounds on $|R_1|,|R_2|,|R_3|$ then gives

$$\begin{aligned}|\pi_s f_s - \pi_u f_u| &\leqslant \|f\|_p\tilde\alpha_{p,1}\left[W^{(p)}(\delta_0,\pi_s)+W^{(p)}(\delta_0,\pi_u)\right]\inf_{t\in\mathbb{R}_+} e^{-Kt} + C_f\tilde\alpha_{p,1}|s-u|^\beta \\ &\quad + C\tilde\alpha^2_{p,1}\frac{M}{K}|s-u|\cdot\|\nabla f\|_p \sup_{\tau\in[0,1]}\sqrt{\bar V(x^\star_\tau)} \\ &\leqslant C|s-u|^\beta\tilde\alpha_p\left[C_f+\tilde\alpha_{p,1}\frac{M}{K}\|\nabla f\|_p\sup_{\tau\in[0,1]}\sqrt{\bar V(x^\star_\tau)}\right].\end{aligned}$$

This completes the proof of (101). For property 2a in the statement, by the Proposition 15 in the time-homogeneous case, for any given s, $f_s\in C^p_2(\mathbb{R}^d)\Rightarrow Q^s_{k\ell}f\in C^p_2(\mathbb{R}^d)$, hence for any $r<+\infty$ and any $\ell\geqslant 0$, $x\mapsto g_{s,r}(x)$ is a member of $C^p_2(\mathbb{R}^d)$. For property 2b in the statement, using (102),

$$|g_{s,\infty}(x)| \leqslant \begin{cases}\ell\|f\|_p\tilde\alpha_p W^{(p)}(\delta_x,\pi_s)\frac{1}{1-e^{-K\ell}}, & \ell>0,\\ \|f\|_p\tilde\alpha_p W^{(p)}(\delta_x,\pi_s)\frac{1}{K}, & \ell=0,\end{cases}$$

which together with Lemma 56 and (58) imply that for any $\ell\geqslant 0$ and $r\in\mathbb{N}_0\cup\{\infty\}$, $\sup_{s,x}|g_{s,\infty}(x)|/(1+\|x\|^{2p+1})<+\infty$. For property 2c, by similar manipulations,

$$|g_{s,\infty}(x)-g_{s,r}(x)| \leqslant \begin{cases}\ell\|f\|_p\tilde\alpha_p W^{(p)}(\delta_x,\pi_s)\frac{e^{-K\ell r}}{1-e^{-K\ell}}, & \ell>0,\\ \|f\|_p\tilde\alpha_p W^{(p)}(\delta_x,\pi_s)\frac{e^{-Kr}}{K}, & \ell=0.\end{cases}$$

For property 2d, in the setting $\ell>0$, with R_1, R_2 and R_3 as above we have

$$g_{u,r}(x)-g_{s,r}(x) = \ell\sum_{k=0}^r R_1(k\ell,x) = (r+1)\ell\big(\pi_s f_s-\pi_u f_u\big) - \ell\sum_{k=0}^r R_2(k\ell,x)+R_3(k\ell,x)$$

and therefore for any $N-1\geqslant r$ for $r\in\mathbb{N}$ and any $N\in\mathbb{N}$ for $r=\infty$

$$\begin{aligned}|g_{s,r}(x)-g_{u,r}(x)| \leqslant & N\ell|\pi_s f_s-\pi_u f_u| + \ell\sum_{k=0}^{N-1}|R_2(k\ell,x)|+|R_3(k\ell,x)| + \ell\sum_{k=N}^{\infty}|R_1(k\ell,x)| \\ \leqslant & C\ell(C_1\vee C_2)\left(N|s-u|^\beta + \frac{e^{-KN\ell}}{1-e^{-K\ell}}\right)\bar V^{(p+1/2)}(x),\end{aligned}$$

with

$$\begin{aligned}C_1 &= \tilde\alpha_{p,1}\left[C_f+\tilde\alpha_{p,1}\frac{M}{K}\|\nabla f\|_p\sup_{\tau\in[0,1]}\sqrt{\bar V(x^\star_\tau)}\right],\\ C_2 &= \|f\|_p\tilde\alpha_{p,1}.\sup_{s\in[0,1]}\pi_s\bar V^{p+1/2}\end{aligned}$$

Clearly

$$C_1\vee C_2 \leqslant C\tilde\alpha_{p,1}(\ell\vee 1)\big(C_f\vee\|f\|_p\big)\Big(1+\tilde\alpha_{p,1}\frac{M}{K}\sup_{\tau\in[0,1]}\sqrt{\bar V(x^\star_\tau)}+\sup_{s\in[0,1]}\pi_s\bar V^{p+1/2}\Big).$$

Now when $|s-u|^\beta \geqslant \frac{e^{-K\ell}}{1-e^{-K\ell}}$, one can choose $N=1$ and conclude. Otherwise we take $N = \lceil -(K\ell)^{-1}\log\big(|s-u|^\beta\big)\rceil$ which with $\beth^{-1} \leqslant 1-K\ell/2$ leads, on the one hand, to

$$\frac{e^{-KN\ell}}{1-e^{-K\ell}} \leqslant \frac{\beth}{K\ell}|s-u|^\beta$$

and on the other hand to

$$N|s-u|^\beta \leqslant \big[1-(K\ell)^{-1}\log\big(|s-u|^\beta\big)\big]|s-u|^\beta$$

So we study $\varphi(x) = x^a \log x$ for $x \geqslant 0$. $\varphi'(x) = x^{a-1}\big[a\log(x)+1\big]$ so $\varphi(x)$ reaches its minimum at $\exp(-a^{-1})$, and therefore since $\varphi(x) \leqslant 0$ for $0 \leqslant x \leqslant 1$, for any $b \geqslant 0$

$$\sup_{x\in[0,b]} \big|\varphi(x)\big| \leqslant \big|\varphi(a)\big| \vee \big|\varphi(b)\big|.$$

Therefore for $|s-u| \leqslant R_f$ and $\zeta \in (0,\beta)$ we have

$$N|s-u|^{\beta-\zeta} \leqslant {R_f}^{\beta-\zeta} + \frac{\beta}{K\ell}\big[e^{-1}/(\beta-\zeta)\big] \vee \big(R_f^{\beta-\zeta}\big|\log R_f\big|\big)$$

and in total we have the bound

$$N|s-u|^\beta + \frac{e^{-KN\ell}}{1-e^{-K\ell}} \leqslant \frac{1}{K\ell}\Big[(2\vee\beth){R_f}^{\beta-\zeta} + \beta\big[e^{-1}/(\beta-\zeta)\big] \vee \big(R_f^{\beta-\zeta}\big|\log R_f\big|\big)\Big]|s-u|^\zeta.$$

For the case $\ell = 0$ a reasoning similar as that above leads to

$$|g_{s,r}(x) - g_{u,r}(x)| \leqslant C\ell(C_1 \vee C_2)\left(N|s-u|^\beta + \frac{e^{-KN}}{K}\right)\bar{V}^{(p+1/2)}(x),$$

□

and for $|s-u|^\beta \geqslant e^{-K}/K$, set $N=1$, and otherwise set $N = \big\lceil -K^{-1}\log\big(|s-u|^\beta\big)\big\rceil$ and deduce from above that

$$\begin{aligned} N|s-u|^\beta + \frac{e^{-KN}}{K} &\leqslant \big[1-K^{-1}\log\big(|s-u|^\beta\big)+K^{-1}\big]|s-u|^\beta \\ &\leqslant \frac{1}{K}\Big[K{R_f}^{\beta-\zeta} + \beta\big[e^{-1}/(\beta-\zeta)\big] \vee \big(R_f^{\beta-\zeta}\big|\log R_f\big|\big)\Big]|s-u|^\zeta \end{aligned}$$

and we conclude by combining all the cases.

Lemma 63. *Assume that for some $p \geqslant 1$ and $f \in C^p_{0,2}([0,1]\times\mathbb{R}^d)$ there exist constants $C_f < +\infty$, $R_f > 0$ and $\beta \in (0,1]$ such that*

$$|s-t| \leqslant R_f \quad \Rightarrow \quad |f_s(x) - f_t(x)| \leqslant C_f|s-t|^\beta \bar{V}^{(p)}(x), \quad \forall x \in \mathbb{R}^d.$$

Then for any $h \in (0,R_f]$

$$\left|h\sum_{k=0}^{\lfloor 1/h\rfloor-1}\pi_{kh}f_{kh} - \int_0^1 \pi_t f_t \mathrm{d}t\right| \leqslant h^\beta\tilde{\alpha}_p\big(C_f \vee \|\nabla f\|_p\big)\left[1+\tilde{\alpha}_p\frac{M}{K}\sup_{t\in[0,1]}\sqrt{\bar{V}(x_t^\star)}\right].$$

Proof. Using Lemma 62,

$$\begin{aligned} &\left|h\sum_{k=0}^{\lfloor 1/h\rfloor-1}\pi_{kh}f_{kh} - \int_0^1 \pi_t f_t \mathrm{d}t\right| \\ &\qquad\leqslant \sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h}|\pi_{kh}f_{kh} - \pi_t f_t|\mathrm{d}t \\ &\qquad\leqslant h^\beta\tilde{\alpha}_p\left[C_f + \tilde{\alpha}_p\frac{M}{K}\|\nabla f\|_p\cdot\sup_{t\in[0,1]}\sqrt{\bar{V}(x_t^\star)}\right]\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h}\mathrm{d}t \\ &\qquad\leqslant h^\beta\tilde{\alpha}_p\left[C_f + \tilde{\alpha}_p\frac{M}{K}\|\nabla f\|_p\cdot\sup_{t\in[0,1]}\sqrt{\bar{V}(x_t^\star)}\right]. \end{aligned}$$

□

7 Controlling the discretization error

Throughout section 7, $(\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$, μ^ϵ , and $\widetilde{\mu}^{\epsilon,h}$ are as defined in section 1.5.3.

7.1 Bounding the total variation distance

Proposition 64. *If $h/\epsilon \in (0, 2K/L^2)$, then for any $\delta \in (0,1)$*

$$\|\mu^\epsilon - \widetilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}} \leqslant \frac{1}{2}\left[L^2 d\frac{h}{\epsilon^2} + \frac{h^3}{3\epsilon}\left(M^2 + \frac{L^4}{\epsilon^2}\right)\left(\frac{1}{h} + \frac{1}{1-\lambda}\left[\mu_0(V_0) + \frac{b}{h}\right]\right)\right]^{1/2},$$

where

$$\begin{aligned}
\lambda &:= 1 - \left(\frac{2hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 L^2\right)(1-\delta), \\
b &:= \sup_{t\in(0,1)} \|\partial_t x_t^\star\|^2 \left[\frac{4h^2}{\delta\left(\frac{2hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 L^2\right)} + h^2\right] + 2d\frac{h}{\epsilon}.
\end{aligned}$$

Proof. The proof is quite similar to [8, Proof of Lemma 2], except that here we need to account for the dependence of U_t on t. Consider

$$\begin{aligned}
\Xi_t &:= \frac{1}{\sqrt{2\epsilon}}\left\{\widetilde{\nabla U}_t(\widetilde{X}_t^{\epsilon,h}) - \nabla U_t(\widetilde{X}_t^{\epsilon,h})\right\} \\
Z_t &:= \exp\left(\sum_{i=1}^d \int_0^t \Xi_s^i \mathrm{d}B_s^i - \frac{1}{2}\int_0^t \|\Xi_s\|^2 \mathrm{d}s\right).
\end{aligned}$$

By Girsanov's theorem, under the probability measure $\widetilde{\mathbb{P}}_{\mathcal{F}_1}[A] := \mathbb{E}[\mathbb{I}_A Z_1]$, $A \in \mathcal{F}_1$, the process $\int_0^t \mathrm{d}B_s - \Xi_s \mathrm{d}s$ is a d-dimensional $(\mathcal{F}_t)_{t\in[0,1]}$-Brownian motion and the law of $(\widetilde{X}_t^{\epsilon,h})_{t\in[0,1]}$ is μ. Denoting by $\mathbb{P}_{\mathcal{F}_1}$ the restriction of $\mathbb{P}$ to $\mathcal{F}_1$, we therefore have by Pinsker's inequality

$$\|\mu^\epsilon - \widetilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}} \leqslant \|\widetilde{\mathbb{P}}_{\mathcal{F}_1} - \mathbb{P}_{\mathcal{F}_1}\|_{\mathrm{tv}} \leqslant \sqrt{-\frac{1}{2}\mathbb{E}[\log Z_1]} = \frac{1}{2}\sqrt{\mathbb{E}\left[\int_0^t \|\Xi_s\|^2 \mathrm{d}s\right]}. \tag{104}$$

For $s \in [kh, (k+1)h)$, we have from (16) and (A2),

$$\begin{aligned}
\mathbb{E}[\|\widetilde{X}_{kh}^{\epsilon,h} - \widetilde{X}_s^{\epsilon,h}\|^2] &= \frac{1}{\epsilon^2}(s-kh)^2 \mathbb{E}[\|\nabla U_{kh}(\widetilde{X}_{kh}^{\epsilon,h})\|^2] + \frac{2d}{\epsilon}(s-kh) \\
&\leqslant \frac{1}{\epsilon^2}(s-kh)^2 L^2 \mathbb{E}[1 + \|\widetilde{X}_{kh}^{\epsilon,h} - x_{kh}^\star\|^2] + \frac{2d}{\epsilon}(s-kh). \qquad (105)
\end{aligned}$$

The considering the expectation in (104), we find from (17), (A5) , (A2), (105), and Lemma 65,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \|\Xi_s\|^2 \mathrm{d}s\right] \\
&= \frac{1}{2\epsilon}\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h} \mathbb{E}[\|\nabla U_{kh}(\widetilde{X}^{\epsilon,h}_{kh}) - \nabla U_s(\widetilde{X}^{\epsilon,h}_s)\|^2]\mathrm{d}s \\
&\leqslant \frac{1}{\epsilon}\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h} \mathbb{E}[\|\nabla U_{kh}(\widetilde{X}^{\epsilon,h}_{kh}) - \nabla U_s(\widetilde{X}^{\epsilon,h}_{kh})\|^2] + \mathbb{E}[\|\nabla U_s(\widetilde{X}^{\epsilon,h}_{kh}) - \nabla U_s(\widetilde{X}^{\epsilon,h}_s)\|^2]\mathrm{d}s \\
&\leqslant \frac{1}{\epsilon}\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h} M^2(s-kh)^2\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2] + L^2\mathbb{E}[\|\widetilde{X}^{\epsilon,h}_{kh} - \widetilde{X}^{\epsilon,h}_s\|^2]\mathrm{d}s \\
&\leqslant \frac{1}{\epsilon}\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h} M^2(s-kh)^2\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2] + L^2\left(\frac{1}{\epsilon^2}(s-kh)^2L^2\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2] + \frac{2d}{\epsilon}(s-kh)\right)\mathrm{d}s \\
&= \frac{1}{\epsilon}\left(M^2+\frac{L^4}{\epsilon^2}\right)\sum_{k=0}^{\lfloor 1/h\rfloor-1}\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2]\int_{kh}^{(k+1)h}(s-kh)^2\mathrm{d}s \\
&\quad + \frac{1}{\epsilon}L^2\frac{2d}{\epsilon}\sum_{k=0}^{\lfloor 1/h\rfloor-1}\int_{kh}^{(k+1)h}(s-kh)\mathrm{d}s \\
&= L^2 d\frac{h}{\epsilon^2} + \frac{h^3}{3\epsilon}\left(M^2+\frac{L^4}{\epsilon^2}\right)\sum_{k=0}^{\lfloor 1/h\rfloor-1}\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2] \\
&\leqslant L^2 d\frac{h}{\epsilon^2} + \frac{h^3}{3\epsilon}\left(M^2+\frac{L^4}{\epsilon^2}\right)\left(\frac{1}{h}+\frac{1}{1-\lambda}\left[\mu_0(V)+\frac{b}{h}\right]\right).
\end{aligned}
$$

Substituting in to (104) completes the proof. □

7.2 Drift condition for the discretized process

Define

$$
\widetilde{P}_k(x,A) := \int_A \frac{1}{\sqrt{4\pi h/\epsilon}}\exp\left(-\frac{1}{4h/\epsilon}\|x - h/\epsilon\nabla U_{kh}(x) - y\|^2\right)\mathrm{d}y,
$$

where the dependence of $\widetilde{P}_k$ on ϵ and h is not shown in the notation.

Lemma 65. *If $h/\epsilon \in (0, 2K/L^2)$, then for any $\delta \in (0,1)$,*

$$
\begin{aligned}
\widetilde{P}_k V_{kh}(x) &\leqslant \lambda V_{(k-1)h}(x) + b, && (106)\\
\sum_{k=0}^{\lfloor 1/h\rfloor-1}\mathbb{E}[1+\|\widetilde{X}^{\epsilon,h}_{kh} - x^\star_{kh}\|^2] &\leqslant \frac{1}{h}+\frac{1}{1-\lambda}\left[\mu_0(V_0)+\frac{b}{h}\right], && (107)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda &:= 1-\left(\frac{2hK}{\epsilon}-\left(\frac{h}{\epsilon}\right)^2 L^2\right)(1-\delta),\\
b &:= \sup_t \|\partial_t x^\star_t\|^2\left[\frac{4h^2}{\delta\left(\frac{2hK}{\epsilon}-\left(\frac{h}{\epsilon}\right)^2L^2\right)}+h^2\right]+2d\frac{h}{\epsilon}.
\end{aligned}
$$

Proof. To simplify presentation in the proof we write $\widetilde{X}_k := \widetilde{X}^{\epsilon}_{kh}$, $x_{k-1} := x_{(k-1)h}$, $x^\star_k := x^\star_{kh}$, $\nabla U_{k-1}(x) := \nabla U_{(k-1)h}(x)$ etc. With $\xi \sim \mathcal{N}(0_d, 2h/\epsilon I_d)$, we have

$$\begin{aligned}\widetilde{P}_k V_{kh}(x) = \mathbb{E}\left[\, \|\widetilde{X}_k - x^\star_k\|^2 \middle| \widetilde{X}_{k-1} = x\right] &= \mathbb{E}\left[\left\| x - \frac{h}{\epsilon}\nabla U_{k-1}(x) + \xi - x^\star_k \right\|^2\right] \\ &\leqslant \left(\left\| x - x^\star_{k-1} - \frac{h}{\epsilon}\nabla U_{k-1}(x)\right\| + \|x^\star_k - x^\star_{k-1}\|\right)^2 + \mathbb{E}[\|\xi\|^2],\end{aligned}$$

where in view of Lemma 68,

$$\|x^\star_k - x^\star_{k-1}\| \leqslant ch, \qquad c := \sup_{t\in(0,1)} \|\partial_t x^\star_t\| < +\infty,$$

and

$$\mathbb{E}[\|\xi\|^2] = 2d\frac{h}{\epsilon}.$$

Now writing $\beta := \frac{2hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 L^2$, noting the assumption $h/\epsilon \in (0, 2K/L^2)$, using (A4) and (A2) we have for any $\delta \in (0,1)$

$$\begin{aligned}&\left\| x - \frac{h}{\epsilon}\nabla U_{k-1}(x) - x^\star_{k-1}\right\|^2 \\ &\leqslant \|x - x^\star_{k-1}\|^2 - \frac{2h}{\epsilon}\left\langle x - x^\star_{k-1}, \nabla U_{k-1}(x)\right\rangle + \left(\frac{h}{\epsilon}\right)^2 \|\nabla U_{k-1}(x)\|^2 \\ &\leqslant (1-\beta)\|x - x^\star_{k-1}\|^2 \\ &= \lambda\|x - x^\star_{k-1}\|^2 - \delta\beta\|x - x^\star_{k-1}\|^2,\end{aligned}$$

where $\lambda := 1 - \beta(1-\delta) < 1$. Combining the above gives:

$$\begin{aligned}\widetilde{P}_k V_{kh}(x) &\leqslant \lambda\|x - x^\star_{k-1}\|^2 - \delta\beta\|x - x^\star_{k-1}\|^2 + 2ch\|x - x^\star_{k-1}\| + c^2h^2 + 2d\frac{h}{\epsilon} \\ &\leqslant \lambda\|x - x^\star_{k-1}\|^2 + \frac{4c^2h^2}{\delta\beta} + c^2h^2 + 2d\frac{h}{\epsilon},\end{aligned}$$

where the final inequality follows by considering whether or not $2ch \leqslant \delta\beta\|x - x^\star_{k-1}\|$. Thus (106) holds and iterating gives

$$\mathbb{E}\left[\|\widetilde{X}_k - x^\star_k\|^2 | X_0 = x\right] \leqslant \lambda^k V_0(x) + b\sum_{j=0}^{k-1}\lambda^j,$$

from which (107) follows. ☐

8 Auxiliary results and proofs

8.1 Preliminaries

Lemma 66.

$$\partial_t \log Z_t = -\int_{\mathbb{R}^d} \partial_t U_t(x) \pi_t(\mathrm{d}x).$$

Proof. Using (A4), Lemma 67, the reverse triangle inequality and the convexity of $a \mapsto a^2$,

$$\begin{aligned}
\sup_t \exp\left[-U_t(x)\right] &\leqslant \sup_t \exp\left[-U_t(x_t^\star) - \frac{K}{2}\|x - x_t^\star\|^2\right] \\
&\leqslant \exp\left[-\inf_t U_t(x_t^\star) - \frac{K}{4}\|x\|^2 + \frac{K}{2}\sup_t \|x_t^\star\|^2\right],
\end{aligned}$$

where $\sup_{t\in[0,1]} \|x_t^\star\|$ and $-\inf_t U_t(x_t^\star)$ are finite, since by Lemma 68, $t \mapsto \|x_t^\star\|$ is continuous on $[0,1]$, and $U_t(x)$ is continous in (t,x) by (A1). Also by (A1), there exists some $p \geqslant 1$ and $c < +\infty$ such that

$$\sup_t |\partial_t U_t(x)| \leqslant c(1 + \|x\|^{2p}), \quad \forall x.$$

Hence the following interchange of differentiation and integration is permitted:

$$\begin{aligned}
\partial_t \log Z_t &= \frac{1}{Z_t}\partial_t \int_{\mathbb{R}^d} \exp\left[-U_t(x)\right] \mathrm{d}x \\
&= -\frac{1}{Z_t}\int_{\mathbb{R}^d} \exp\left[-U_t(x)\right] \partial_t U_t(x) \mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \partial_t U_t(x) \pi_t(\mathrm{d}x).
\end{aligned}$$

$\square$

Lemma 67. *For any given $f \in C_2(\mathbb{R}^d)$ and $c > 0$, the following conditions are equivalent:*

$$\begin{aligned}
f(y) - f(x) &\geqslant \langle \nabla f(x), y - x\rangle + \frac{1}{2}c\|y - x\|^2, \quad \forall x, y \in \mathbb{R}^d, \\
\langle \nabla f(x) - \nabla f(y), x - y\rangle &\geqslant c\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^d, \\
\inf_{x\in\mathbb{R}^d} \sum_{i,j} v_i \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_j &\geqslant c\|v\|^2, \quad \forall v \in \mathbb{R}^d.
\end{aligned}$$

Proof. See [29]. $\square$

Lemma 68. *Let $x_t^\star$ be the unique minimizer of U_t. Then the map $t \mapsto x_t^\star$ is continuous on $[0,1]$, continously differentiable on $(0,1)$ and*

$$\sup_{t\in(0,1)} \|\partial_t x_t^\star\| \vee \sup_{t\in[0,1]} \|x_t^\star\| \leqslant \frac{M}{K}.$$

Proof. Fix any $t \in (0,1)$. The strong convexity assumption (A4) implies $\nabla^{(2)} U_t(x)$ is invertible for all x. Therefore by the implicit function theorem there exist open neighborhoods $\mathcal{T}$ of t and $\mathcal{X}$ of $x_t^\star$ and a unique continuously differentiable function $\zeta : \mathcal{T} \to \mathcal{X}$ such that $\{(s, \zeta(s))\,;\, s \in \mathcal{T}\} = \{(s,x)\,;\, \nabla U_s(x) = 0, (s,x) \in \mathcal{T} \times \mathcal{X}\}$. Since $t \in (0,1)$ was arbitrary, the interval $(0,1)$ can be covered with such neighborhoods $\mathcal{T}$, and the uniqueness under (A4) of the minimizer $U_t(\cdot)$ for each t implies that the continuously differentiable functions must agree on the non-empty intersections between the $\mathcal{T}$'s, yielding a continuously differentiable function $\zeta : (0,1) \to \mathbb{R}^d$ such that $\zeta(t) = x_t^\star$. Let us now prove that $\lim_{t\searrow 0} \zeta(t) = x_0^\star$. First note that ∇U_t is continuous in t on $[0,1]$ by assumption, so $\lim_{n\to+\infty} \|\nabla U_{n^{-1}}(x_0^\star)\| = \|\nabla U_0(x_0^\star)\| = 0$. By way of a contradiction, suppose that there exists $\delta > 0$ such that for all $n_0 > 0$ there exists $n \geqslant n_0$ such that

$$\|x_0^\star - \zeta(n^{-1})\| \geqslant \delta,$$

which together with (A4), Lemma 67 and Cauchy-Schwartz implies

$$\begin{aligned}\|\nabla U_{n^{-1}}(x_0^\star)\| &= \|\nabla U_{n^{-1}}(x_0^\star) - \nabla U_{n^{-1}}(\zeta(n^{-1}))\| \\ &\geqslant K\|x_0^\star - \zeta(n^{-1})\| \geqslant K\delta,\end{aligned}$$

giving a contradiction as required. By a similar argument $\lim_{t\nearrow 1}\zeta(t) = x_1^\star$, and therefore $t \mapsto x_t^\star$ is continuous on $[0,1]$.

We also have:

$$\|\partial_t x_t^\star\| = \left\|[\nabla^{(2)}U_t]^{-1}(x_t^\star)\cdot \partial_t \nabla U_t(x)|_{x=x_t^\star}\right\| \leqslant \frac{1}{K}\left\|\partial_t \nabla U_t(x)|_{x=x_t^\star}\right\|, \tag{108}$$

where the equality is due to the implicit function theorem and the inequality uses the facts that: for a symmetric matrix H, the operator norm $\|H\|_{\mathrm{op}}$ induced by the Euclidean distance on $\mathbb{R}^d$ is equal to the largest eigenvalue of H; $\|H^{-1}x\| \leqslant \|H^{-1}\|_{\mathrm{op}}\|x\|$; and (A4) implies all the eigenvalues of $\nabla^{(2)}U_t(x)$ are lower bounded by K. The term on the right of (108) is uniformly bounded over $t \in (0,1)$ by K/M because (A5) implies

$$\|\nabla U_t(x_t^\star) - \nabla U_{t+\delta}(x_t^\star)\| \leqslant M\delta.$$

Integrating this bound and noting that $x_0^\star = 0$ by (A4),

$$\sup_{t\in[0,1]}\|x_t^\star\| \leqslant \|x_0^\star\| + \sup_{t\in[0,1]}\int_0^t \|\partial_s x_s^\star\|\mathrm{d}s \leqslant \frac{M}{K}.$$

□

Lemma 69. *For any $p \geqslant 1$, $t \in [0,1]$ and $f \in C_0^p(\mathbb{R}^d)$,*

$$\mathrm{var}_{\pi_t}[f] \geqslant L^{-1}\sum_{i=1}^d \pi_t\left(f\frac{\partial U_t}{\partial x_i}\right)^2.$$

Proof. Fix any $t \in [0,1]$ and $f \in C_0^p(\mathbb{R}^d)$. The first part of the proof follows arguments used to derive Cramer-Rao inequalties, see [4] for perspective on this kind of technique. Let Θ be any compact subset of $\mathbb{R}^d$ containing 0, and then introduce an artificial location parameter $\theta \in \Theta$. Suppressing t to simplify notation, consider the probability measure π^θ defined by

$$\pi^\theta(\mathrm{d}x) := \pi^\theta(x)\mathrm{d}x, \quad \pi^\theta(x) := Z_t^{-1}\exp\{-U^\theta(x)\}\mathrm{d}x, \quad U^\theta(x) := U_t(x-\theta).$$

Then with expectation and variance with respect to π^θ denoted respectively by $\mathbb{E}^\theta[\cdot]$ and $\mathrm{var}^\theta[\cdot]$, and gradient with respect to θ denoted by ∇_θ, define the vector $g_\theta := \nabla_\theta \mathbb{E}^\theta[f(X)]$ and the matrix $J_\theta := -\mathbb{E}^\theta[\nabla_\theta^{(2)}\log \pi^\theta(X)]$, where in the latter and similar expressions below, the expectation is element-wise. Using (A4), (A2), (A3) and Lemma 68, it can be checked using manipulations similar to those in the proof of Lemma 66 that the following identities hold by differentiation under the integral sign:

$$\begin{aligned} g_\theta &= \mathbb{E}^\theta[f(X)\nabla_\theta \log \pi^\theta(X)], \\ 0 &= \mathbb{E}^\theta[\nabla_\theta \log \pi^\theta(X)], \\ J_\theta &= \mathbb{E}^\theta[\nabla_\theta \log \pi^\theta(X)\cdot\{\nabla_\theta \log\pi^\theta(X)\}^T], \end{aligned}$$

and J_θ is invertible. Using these identities and Cauchy-Schwartz,

$$\begin{aligned} g_\theta^T J_\theta^{-1} g_\theta &= g_\theta^T J_\theta^{-1}\mathbb{E}^\theta[f(X)\nabla_\theta \log \pi^\theta(X)] \\ &= g_\theta^T J_\theta^{-1}\mathbb{E}^\theta[\{f(X) - \mathbb{E}^\theta[f(X)]\}\nabla_\theta \log\pi^\theta(X)] \\ &= \mathbb{E}^\theta[\{f(X) - \mathbb{E}^\theta[f(X)]\} g_\theta^T J_\theta^{-1}\nabla_\theta \log \pi^\theta(X)] \\ &\leqslant \mathrm{var}^\theta[f(X)]^{1/2}\mathbb{E}^\theta[(g_\theta^T J_\theta^{-1}\nabla_\theta \log\pi^\theta(X))^2]^{1/2} \\ &= \mathrm{var}^\theta[f(X)]^{1/2}(g_\theta^T J_\theta^{-1} g_\theta)^{1/2}, \end{aligned}$$

hence

$$\mathrm{var}^{\theta}[f(X)] \geqslant g_\theta^T J_\theta^{-1} g_\theta. \tag{109}$$

Noting that $\nabla_\theta \log \pi^\theta(x) = \nabla U(x-\theta)$ and $\nabla_\theta^{(2)} \log \pi^\theta(x) = -\nabla^{(2)} U(x-\theta)$, the lower bound (109) with $\theta = 0$ reads:

$$\mathrm{var}_\pi[f] \geqslant \mathbb{E}_\pi[f\nabla U]^T \mathbb{E}_\pi[\nabla^{(2)} U]^{-1} \mathbb{E}_\pi[f\nabla U]. \tag{110}$$

Using Cauchy-Schwartz and the Lipschitz assumption (A2), we have for any $\tau > 0$ and $v \in \mathbb{R}^d$

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau \left\langle \nabla^{(2)} U(x+\lambda v)\cdot v, v\right\rangle \mathrm{d}\lambda &= \frac{1}{\tau}\langle \nabla U(x+\tau v) - \nabla U(x), v\rangle \\ &\leqslant \frac{1}{\tau}\|\nabla U(x+\tau v) - \nabla U(x)\|\|v\| \\ &\leqslant L\|v\|^2.\end{aligned}$$

Taking $\tau \to 0$ we find $v^T \mathbb{E}_\pi[\nabla_x^{(2)} U]v \leqslant L\|v\|^2$, so $v^T \mathbb{E}_\pi[\nabla_x^{(2)} U]^{-1} v \geqslant L^{-1}\|v\|^2$, which applied to (110) completes the proof. □

8.2 Intermediate results concerning dimension dependence

Lemma 70. *Fix $p \geqslant 1$ and consider the quantities α_p and $\tilde{\alpha}_p$ defined in Lemmas 14 and 61, choosing there $\kappa = Kp/2$.*
1) $\tilde{\alpha}_p$ does not depend on ϵ. For any $q \geqslant 0$, if $K^{-1} \vee \sup_t \|x_t^\star\|^2 = O(d^q)$ as $d \to \infty$, then $\tilde{\alpha}_p = O(d^{p(q+1)})$.
2) For any $q \geqslant 0$, if $K^{-1} \vee \sup_t \|x_t^\star\|^2 = O(d^q)$ and $\frac{\epsilon}{K}\sup_t \|\partial_t x_t^\star\| = O(1)$ as $d \to \infty$, then $\alpha_p = O(d^{p(q+1)})$.

Proof. For part 1) the expression for $\tilde{\alpha}_p$ in Lemma 61 with κ chosen to be $Kp/2$ is:

$$\begin{aligned}\tilde{\alpha}_p &= 2^{4p-2} \vee \left[1 + 2^{2p-1}\left(\frac{4}{K^p}(8(p-1)+4d)^{p-1}[2(p-1)+d] + (1+2^{2p-1})\sup_t \|x_t^\star\|^{2p}\right)\right] \\ &= O\left(1 + \frac{d^p}{K^p} + \sup_t \|x_t^\star\|^{2p}\right),\end{aligned}$$

from which the second claim of part 1) follows.

For part 2), writing out the expression for α_p from Lemma 14 with $\kappa = Kp/2$ and the shorthand $v := \sup_t \|\partial_t x_t^\star\|$,

$$\alpha_p = 2^{4p-2} \vee \left[1 + 2^{2p-1}\left(\frac{4}{K} r^{2p-2}\left[r\epsilon v + [2(p-1)+d]\right] + (1+2^{2p-1})\sup_t \|x_t^\star\|^{2p}\right)\right]$$

where

$$r = \frac{\epsilon v}{K} + 2\sqrt{\frac{\epsilon^2 v^2}{K^2} + \frac{1}{K}[2(p-1)+d]}.$$

Using the hypotheses of part 2), we find $r = O(1 + \sqrt{1+d/K}) = O(\sqrt{d/K})$, and so

$$\begin{aligned}\alpha_p &= O\left(r^{2p-2}\left(r\frac{\epsilon v}{K} + \frac{d}{K}\right) + \sup_t \|x_t^\star\|^{2p}\right) \\ &= O\left(\left(\frac{d}{K}\right)^{(p-1)}\left(r + \frac{d}{K}\right) + \sup_t \|x_t^\star\|^{2p}\right) \\ &= O\left(\frac{d^p}{K^p} + \sup_t \|x_t^\star\|^{2p}\right) \\ &= O(d^{p(q+1)}).\end{aligned}$$

□

Lemma 71. *Fix $p \geqslant 1$. For any $q \geqslant 0$, if $K^{-1} \vee \sup_{t\in[0,1]} \|x_t^\star\|^2 = O(d^q)$ as $d \to \infty$, then $\sup_{t\in[0,1]} \pi_t(\bar{V}^{(p)}) = O(d^{p(q+1)})$.*

Proof. We have

$$\pi_t(\bar{V}^{(p)}) \leqslant 1 + 2^{2p-1}\pi_t(V_t^p) + 2^{2p-1}\|x_t^\star\|^{2p}. \tag{111}$$

By an application of (99) with there $\epsilon = 1$ and $\kappa = Kp/2$, we have for any $s > 0$,

$$\pi_t(V_t^p) = \pi_t Q_s^{t,1} V_t^p \leqslant e^{-\delta s}\pi_t(V_t^p) + \frac{\tilde{b}}{\delta},$$

where

$$\tilde{r} = 2\sqrt{\frac{2(p-1)+d}{K}}, \quad \tilde{b} = 2p\tilde{r}^{2(p-1)}(2(p-1)+d), \quad \delta = Kp/2.$$

hence taking $s \to \infty$, we obtain under the hypothesis $K^{-1} = O(d^q)$,

$$\begin{aligned}\sup_t \pi_t(V_t^p) &\leqslant \frac{\tilde{b}}{\delta} = \frac{4}{K}2^{2(p-1)}\left(\frac{2(p-1)+d}{K}\right)^{(p-1)}(2(p-1)+d) \\ &= O\left(\frac{1}{K}\left(\frac{d}{K}\right)^{p-1} d\right) = O\left(\frac{d^p}{K^p}\right) = O(d^{p+pq}),\end{aligned}$$

and combining this with (111) and the hypothesis $\sup_t \|x_t^\star\|^2 = O(d^q)$ completes the proof. ◻

Lemma 72. *For any $q \geqslant 0$, if*

$$K^{-1} \vee \sup_t \|\partial_t x_t^\star\|^2 \vee \sup_t \|x_t^\star\|^2 = O(d^q), \quad \mu_0(V) = O(d^{q+1}),$$

$$h \vee \epsilon \vee \frac{h}{\epsilon}\frac{L^2}{K} = o(1), \quad \frac{h}{\epsilon^2}d^{3q} = O(1),$$

as $d \to \infty$, then

$$h\sum_{k=0}^{\lfloor 1/h\rfloor -1} 1 + \mathbb{E}[\|\widetilde{X}_{kh}^{\epsilon,h}\|^2] = O(\epsilon d^{2q+1} + hd^{q+1} + d^q).$$

Proof. We have

$$h\sum_{k=0}^{\lfloor 1/h\rfloor -1} 1 + \mathbb{E}[\|\widetilde{X}_{kh}^{\epsilon,h}\|^2] \leqslant 2h\sum_{k=0}^{\lfloor 1/h\rfloor -1} 1 + \mathbb{E}[\|\widetilde{X}_{kh}^{\epsilon,h} - x_{kh}^\star\|^2] + 2h\sum_{k=0}^{\lfloor 1/h\rfloor -1}\|x_{kh}^\star\|^2. \tag{112}$$

To estimate the first term on the r.h.s. of (112), consider Lemma 65 with δ there chosen to be 1/2 and note that under the hypothesis $\frac{h}{\epsilon}\frac{L^2}{K} = o(1)$, we have $h/\epsilon \in (0, 2K/L^2)$ for all d large enough. For any such d, the bound of (107) written out explicitly together with the hypotheses $K^{-1} \vee \sup_t \|\partial_t x_t^\star\|^2 = O(d^q)$, $\mu_0(V) = O(d^{q+1})$ and $\frac{h}{\epsilon^2}d^{3q} = O(1)$, $h \vee \epsilon = o(1)$ then gives

$$\begin{aligned}&h\sum_{k=0}^{\lfloor 1/h\rfloor -1} 1 + \mathbb{E}[\|\widetilde{X}_{kh}^{\epsilon,h} - x_{kh}^\star\|^2] \\ &\leqslant 1 + \frac{h}{\frac{hK}{\epsilon}\left(1 - \frac{1}{2}\frac{h}{\epsilon}\frac{L^2}{K}\right)}\left[\mu_0(V) + \sup_t\|\partial_t x_t^\star\|^2 h^2\left\{\frac{4}{\frac{hK}{\epsilon}\left(1 - \frac{1}{2}\frac{h}{\epsilon}\frac{L^2}{K}\right)} + 1\right\} + 2d\frac{h}{\epsilon}\right] \\ &= O\left(1 + \frac{\epsilon}{K}\left[d^{q+1} + d^q h^2\left\{\frac{\epsilon}{hK} + 1\right\} + d\frac{h}{\epsilon}\right]\right) \\ &= O\left(1 + \epsilon d^q\left[d^{q+1} + d^{2q}h\epsilon + d^q h^2 + d\frac{h}{\epsilon}\right]\right) \\ &= O\left(1 + \epsilon d^{2q+1} + d^{3q}h\epsilon^2 + d^{2q}h^2\epsilon + d^{q+1}h\right) \\ &= O\left(1 + \epsilon d^{2q+1} + d^{q+1}h\right).\end{aligned}$$

The proof is completed by combining this estimate with the fact that the second term on the r.h.s. of (112) is in $O(d^q)$ due to the hypothesis $\sup_t \|x_t^\star\|^2 = O(d^q)$. ◻

Proof of Proposition 10. First note that the hypothesis $\frac{h}{\epsilon}\frac{L^2}{K} \in o(1)$ implies that for d large enough, $h/\epsilon \in (0, 2K/L^2)$. Then for such d and choosing $\delta = 1/2$ in Proposition 64, we have

$$\begin{aligned}
&\|\mu^\epsilon - \tilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}}^2 \\
&\leqslant L^2 d\frac{h}{\epsilon^2} + \frac{h^3}{3\epsilon}\left(M^2 + \frac{L^4}{\epsilon^2}\right) \\
&\quad \cdot \left(\frac{1}{h} + \frac{1}{1-\lambda}\left[\mu_0(V_0) + \frac{1}{h}\left(\sup_{t\in(0,1)} \|\partial_t x_t^\star\|^2 \left[\frac{4h^2}{\delta\left(\frac{2hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 L^2\right)} + h^2\right] + 2d\frac{h}{\epsilon}\right)\right]\right) \\
&= L^2 d\frac{h}{\epsilon^2} + \frac{1}{3}\left(hM^2 + \frac{h}{\epsilon^2}L^4\right) \\
&\quad \cdot \left(\frac{h}{\epsilon} + \frac{\frac{h}{\epsilon}}{\frac{hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 \frac{L^2}{2}}\left[h\mu_0(V_0) + \epsilon h \sup_{t\in(0,1)} \|\partial_t x_t^\star\|^2 \left[\frac{4\frac{h}{\epsilon}}{\left(\frac{2hK}{\epsilon} - \left(\frac{h}{\epsilon}\right)^2 \frac{L^2}{2}\right)} + \frac{h}{\epsilon}\right] + 2d\frac{h}{\epsilon}\right]\right) \\
&= L^2 d\frac{h}{\epsilon^2} + \frac{1}{3}\left(hM^2 + \frac{h}{\epsilon^2}L^4\right) \\
&\quad \cdot \left(\frac{h}{\epsilon} + \frac{1}{K - \frac{h}{\epsilon}\frac{L^2}{2}}\left[h\mu_0(V_0) + \epsilon h \sup_{t\in(0,1)} \|\partial_t x_t^\star\|^2 \left[\frac{4}{K - \frac{h}{\epsilon}\frac{L^2}{2}} + \frac{h}{\epsilon}\right] + 2d\frac{h}{\epsilon}\right]\right).
\end{aligned}$$

Using the hypotheses (18), $\frac{h}{\epsilon}L^2/K = o(1)$, , $dh/\epsilon = O(1)$, $h = o(1)$, and $\epsilon = o(1)$, we obtain

$$\begin{aligned}
\|\mu^\epsilon - \tilde{\mu}^{\epsilon,h}\|_{\mathrm{tv}}^2 &= O\left(d^{q/2+1}\frac{h}{\epsilon^2} + \left(hd^q + \frac{h}{\epsilon^2}d^q\right)\left(\frac{h}{\epsilon} + d^q\left[hd^{q+1} + \epsilon h d^q[d^q + \frac{h}{\epsilon}] + d\frac{h}{\epsilon}\right]\right)\right) \\
&= O\left(d^{q/2+1}\frac{h}{\epsilon^2} + \left(hd^q + \frac{h}{\epsilon^2}d^q\right)\left(\frac{h}{\epsilon} + hd^{2q+1} + \epsilon h d^{3q} + d^{q+1}\frac{h}{\epsilon}\right)\right) \\
&= O\left(d^{q/2+1}\frac{h}{\epsilon^2} + \left(\frac{h^2}{\epsilon}d^q + h^2 d^{3q+1} + \epsilon h^2 d^{4q} + \frac{h^2}{\epsilon}d^{2q+1}\right) + \left(\frac{h^2}{\epsilon^3}d^q + \frac{h^2}{\epsilon^2}d^{3q+1} + \frac{h^2}{\epsilon}d^{4q} + \frac{h^2}{\epsilon^3}d^{2q+1}\right)\right) \\
&= O\left(\left[\epsilon h^2 + \frac{h^2}{\epsilon}\right]d^{4q} + \left[h^2 + \frac{h^2}{\epsilon^2}\right]d^{3q+1} + \left[\frac{h^2}{\epsilon} + \frac{h^2}{\epsilon^3}\right]d^{2q+1} + \left[\frac{h^2}{\epsilon} + \frac{h^2}{\epsilon^3}\right]d^q + \frac{h}{\epsilon^2}d^{q/2+1}\right) \\
&= O\left(\frac{h^2}{\epsilon}d^{4q} + \frac{h^2}{\epsilon^2}d^{3q+1} + \frac{h^2}{\epsilon^3}d^{2q+1} + \frac{h}{\epsilon^2}d^{q/2+1}\right) \\
&= O\left(\frac{h}{\epsilon^2}d^{4q}\left[\epsilon h + hd^{1-q} + \frac{h}{\epsilon}d^{1-2q} + d^{1-7q/8}\right]\right) \\
&= O\left(\frac{h}{\epsilon^2}d^{4q+1}\right).
\end{aligned}$$

Taking the square root completes the proof. ◻

Lemma 73. *Fix $p \geqslant 1$ and for each $d \in \mathbb{N}$, $f \in C^p_{1,2}([0,1]\times\mathbb{R}^d)$. Assume that (A7) holds and that $\sup_s \|\tilde{\mathcal{L}}_s f_s\|_{p+1/2}$, grows at most polynomially fast as $d \to \infty$, where $\tilde{\mathcal{L}}_s f_s = -\langle \nabla U_s, \nabla f_s\rangle + \Delta f_s$. If $\sup_{t\in[0,1]} 1/\mathrm{var}_{\pi_t}[f_t]$ grows at most polynomially fast as $d \to \infty$, then for any $\ell \geqslant 0$ so does $\sup_{t\in[0,1]} 1/\varsigma_\ell(t)$.*

Proof. We first address the case $\ell = 0$. Using the formula (96), we have

$$\varsigma_0(s) = \int_0^\infty \rho_s(t)\mathrm{d}t,$$

where assuming w.l.o.g. that f_t is centrered with respect to π_t, $\rho_s(t) := \pi_s(f_s Q_t^s f_s)$. Due to the reversibility of Q_t^s with respect to π_s, $\rho_s(t)$ is a nonnegative, therefore for any $r \geqslant 0$

$$\varsigma_0(s) \geqslant \int_0^r \rho_s(t)\mathrm{d}t. \tag{113}$$

We shall now show that

$$\sup_s |\rho_s(0) - \rho_s(t)| \leqslant tC(d), \tag{114}$$

where $C(d)$, to be identified below, grows at most polynomially fast with d. To this end, note that

$$|\rho_s(0) - \rho_s(t)| \leqslant \pi_s(|f_s||(Id - Q_t^s)(f_s)|)$$

and by the time-homogeneous counterpart of Proposition 16,

$$|(Q_t^s - Id)(f_s)|(x) = \left|\int_0^t \partial_u Q_u^s f_s(x)\mathrm{du}\right| = \left|\int_0^t Q_u \tilde{\mathcal{L}}_s f_s(x)\mathrm{du}\right| \leqslant t\|\tilde{\mathcal{L}} f_s\|_{p+1/2}\tilde{\alpha}_{p+1/2}\bar{V}^{(p+1/2)}(x),$$

where $\tilde{\alpha}_{p+1/2}$ is as in Proposition 61 with κ there chosen to be $Kp/2$, and we note that $\|\mathcal{L}_s f_s\|_{p+1/2}$ is finite by Proposition 15. We therefore have

$$|\rho_s(0) - \rho_s(t)| \leqslant t\|\mathcal{L}_s f_s\|_{p+1/2}\tilde{\alpha}_{p+1/2}\pi_s(\bar{V}^{(p)}\bar{V}^{(p+1/2)}),$$

and (114) holds as claimed with $C(d) := \tilde{\alpha}_{p+1/2}\sup_s \|\mathcal{L}_s f_s\|_{p+1/2}\sup_s \pi_s(\bar{V}^{(p)}\bar{V}^{(p+1/2)})$, which indeed grows at most polynomially with d by the hypotheses of the lemma, Lemma 70 and Lemma 71.

Returning then to (113) and applying (114), we otbain

$$\frac{1}{\varsigma_0(t)} \leqslant \frac{1}{r\rho_t(0)}\frac{1}{\left(1 - \frac{rC(d)}{2\rho_t(0)}\right)}.$$

Noting the hypothesis of the lemma on $\sup_t 1/\mathrm{var}_{\pi_t}[f_t]$, and that $\rho_t(0) = \mathrm{var}_{\pi_t}[f_t]$, the proof is completed by choosing $r = d^{-a}$ for $a > 0$ large enough.

The case $\ell > 0$ is more straightforward, since in that situation by (96) and the reversibility of Q_t^s, $\varsigma_\ell(s) \geqslant \ell\mathrm{var}_{\pi_s}[f_s]$. □

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